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THEORY

OF

VOUSSOIR ARCHES.

BY

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THIRD EDITION, REVISED AND ENLARGED.

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1905.

PREFACE.

In the present edition of this work, the method of drawing trial lines of resistance, given in the first edition, is retained and several new chapters added containing discussions of new constructions, equilibrium polygous, properties of curves of resistance, unit stresses, and in the last two chapters, two independent developments of the application of the theory pertaining to solid arches "fixed at the ends" to voussoir arches. The Appendix contains the discussion of the experiments on wooden arches, at the limits of stability, given in the former edition, though here the matter is presented in a more condensed form.

It will probably be admitted that the theory of solid arches, fixed at the ends, applies directly to voussoir arches, when no mortar is used in the joints and the voussoirs fit perfectly between the skew-

backs when not under stress, the resistance of the backing to distortion of the arch under stress being neglected and all loads being supposed to be transmitted vertically to the arch ring.

If the line of resistance determined from this theory, for the original joints, everywhere lies within the middle third of the arch ring, no further trial, on the supposition of other bearing joints, has to be made, so as to make the assumed and computed joints agree.

The investigation in this treatise is limited to this case.

The theory is likewise approximately applicable where thin layers of cement mortar are interposed between some or all of the arch stones that are allowed to harden well before the centres are struck.

For such cases, the design of a number of arches, for spans from 0 to 160 feet, for a rise of one-fifth the span, are given in Chapter IV. and the attention of constructors is particularly called to this table and the comparison of results with certain empirical formulas as shown in figure 25.

It has long been the opinion of the author that the much heavier locomotive loads of to-day require greater depths of keystone than are given by many formulas in current use, and he submits that the results of Chapters IV. and V. effectually establish this position and show the danger of ignoring a theoretical treatment of the subject even where only approximately applicable, as in the case of arches as actually built. The loads may not be transmitted vertically and the spandrel filling may resist spreading, and, in fact, act partly as an arch; but, it is better not to count on these extra elements of stability, except as neutralizing the *dynamic* effect of the moving load alone, and *the middle third limit* prescribed may be looked upon in the light of introducing a factor of safety against the effect of the loads regarded as *static.*

The author has derived assistance in the theory from Scheffler's Théorie des Voutes; also from Prof. Greene's "Arches," and Prof. Eddy's "New Constructions in Graphical Statics."

He has endeavored to assign credit at the proper places in the text to these and other authors.

CHAPEL HILL, May 3d, 1893.

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THEORY OF VOUSSOIR ARCHES.

CHAPTER I.

1. The voussoir arch is composed of blocks of stone, brick or other material, in the shape of truncated wedges, called *arch stones* or *voussoirs*, whose inferior surface, known as the *soffit*, is usually cylindrical and perpendicular to the radiating surfaces of contact called the joints. Between the stones, mortar, either common or preferably hydraulic, is generally interposed.

The lowest voussoirs rest against the *abutments* along the surfaces called the *skewbacks* or *springing joints* and the lower edge of these joints is called the *springing line.*

Figure 1 represents a section of the arch made by a plane perpendicular to a generating element of the cylindrical surface. This plane intersects the soffit in a curved line called the *intrados* and the exterior

surface of the arch ring in a line called the *extrados.* The intrados is generally an arc of a circle or two arcs of circles as in the gothic arch, an ellipse or a false ellipse (basket-handle) composed of several arcs of circles tangent to each other.

The *crown* of the arch is the highest

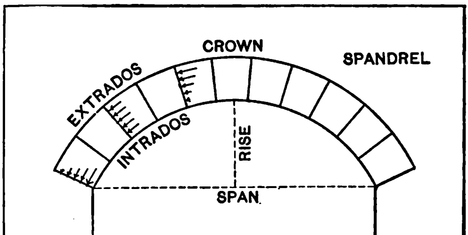


Fig. 1

part of it, the *crown joint* is an imaginary vertical joint through the crown.

The *keystone* or key is the highest arch stone, extending generally either side of the crown, so that there is no *actual* joint at the crown.

The *haunch* or *reins* is the part of the arch between the crown and skewback, and the *spandrel* is the part of the structure

above the extrados and limited by the roadway or other superior surface. The arch is limited at the ends or *heads*, by planes called *faces*—perpendicular to a generating element of the soffit in *right cylindrical arches*, which alone will be considered in this treatise. The *spandrel walls* at the faces are usually of a superior quality of masonry, but the space between them is often filled with inferior masonry, gravel or earth, called the *spandrel filling* or simply the *backing*.

In large arches this space is often occupied by separate walls, running parallel with the roadway and connected by flat stones or light arches, which support the material of the roadway.

The *span* of the arch (Figure 1) is the perpendicular distance between the springing lines, and the *rise* is the vertical distance from the plane of the springing lines to the highest part of the intrados.

The *axis* or *axes* of the cylindrical surface of the soffit will be called likewise the axis or axes of the arch.

When the radial length of joint is the

same throughout, the arch ring is said to be of *uniform section* and its constant *depth*, measured radially, is the depth of keystone.

2. In investigating the strength and stability of right cylindrical arches, it is convenient to consider a slice of the arch and load comprised between two vertical planes, perpendicular to the axis or axes of the arch, and one foot apart, and it will be understood that all subsequent figures refer to such a slice whether distinctly stated or not.

As the spandrel filling between the end walls offers less resistance to deformation of the arch ring than the spandrel walls at the faces, it is proper to consider the slice to be taken in the interior, where the backing is of the least resisting character, in order that we may investigate the most unfavorable case.

As the external forces, including the weight of arch, will be considered symmetrical with respect to the vertical plane midway between the faces of the slice, all the forces may be considered as acting in

the medial plane. The same evidently obtains for the internal stresses at the joints, so that we have simply to investigate the equilibrium of forces in one plane.

The hypothesis will be made that the weight per cubic foot of the arch stones is the same throughout; similarly for the spandrel filling. This is not exactly true, but is certainly near enough, when stones are selected from the same quarry, especially when the arch is investigated as to the action of the very heavy eccentric loads of to-day; for any irregularity can be supposed included in the hypothetical loading (never exactly realized) with perfect safety.

Other approximations will be carefully noted as we proceed.

3. In constructing voussoir arches a wooden frame or centre is built between the abutments, whose upper convex surface exactly coincides with the soffit of the arch to be built. The lower arch stones are laid first, and it is a matter of experience that in a semi-circular arch the arch

stones can be laid successively up to about half the rise without any centering whatsoever. This corresponds to the joint which makes an angle of 30° with the horizontal, or 60° with the vertical. The central portion of 120° has to be supported by the centre, until finally the keystone (which should exactly fit the remaining space) is driven in and the centre removed.

If the spandrel wall is built up to the 30° joint as solidly as the abutment, as should always be done, the arch and spandrel below this joint can be considered as rigid and a part of the abutments; hence the true arch is the central portion of 120° total amplitude.

On this account it is convenient to define a *segmental arch* as one whose total amplitude, or angle between the planes of its extreme joints, is equal to or less than 120°.

The pressure of one arch stone against another at the joints gives rise to molecular stresses, represented by the little arrows in figure 1. It is our object first to find the resultants of these stresses on each joint.

The thrust at the crown is thus the resultant of the stresses on one side of the crown joint exerted against and balanced by the stresses on the other side of the joint. Similarly for any other joints.

Symmetrical Arches.

4. We shall consider first an arch formed of two branches AC, BC (Figure 2), symmetrical and placed in juxtaposition, and comprised between two parallel vertical planes perpendicular to the axis of the arch, the arch being right cylindrical. This arch, composed of voussoirs in the shape of wedges, leans against two abutments at its extremities A and B, and is loaded not only with its own weight but with any other weight whatsoever, distributed symmetrically on either side of the crown C. The mass of the arch is subject to the laws of friction in its joints. The adherence of the mortar, interposed between the voussoirs, being difficult to estimate will not be considered. As the two half arches are symmetrical as to the crown C, it is clear that the points of

application, A and B of the reactions R_1 and R_2 of the surfaces of support, will be also symmetrical in relation to the vertical

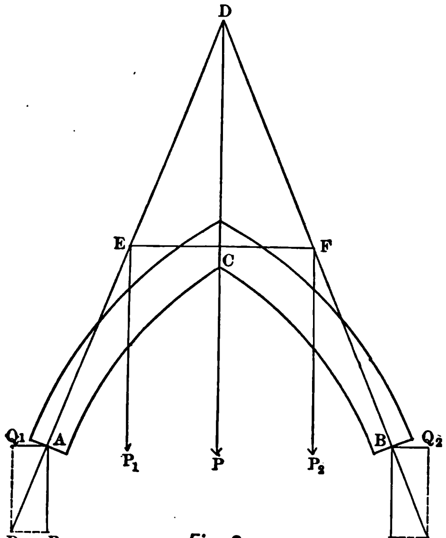


Fig. 2

passing through the crown, and that the line AB will be horizontal, in whatsoever manner the points A and B may vary upon the surfaces of support.

Now the weight of arch and load $= P$, acting downwards through the crown C, together with the reactions R_1 and R_2 acting upwards, form a system of forces in equilibrium, and because R_1 and R_2 must meet P in a point D they are equally inclined to P and hence are equal.

If we decompose the reactions, R_1 and R_2, into their horizontal and vertical components P_1, Q_1, P_2, Q_2, we should have $P_1 = P_2 =$ the weight of half arch with its load, and the thrust $Q_1 = Q_2$.

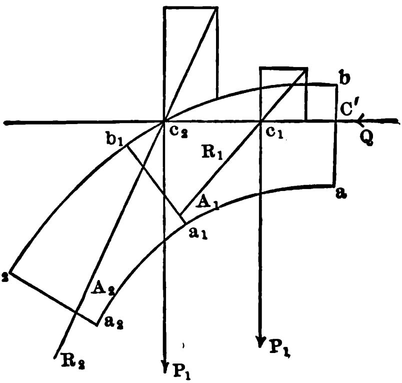
Let us consider now one of the two halves, for example AC. Let EP_1 be the vertical passing through the centre of gravity of this half with its load; to hold this mass in equilibrium it is necessary that there exists at the crown a force whose direction CE passes through the point of intersection E of the vertical EP_1 with the direction of the reaction R_1.

As the vertical component of R_1 equals the weight P_1 acting through E and since by the laws of statics, the algebraic sum of the horizontal components of the forces acting on the half arch AC, must equal

zero, the thrust at the crown C of the arch is necessarily equal to the second component Q_1, of the reaction R. and must be horizontal as it is.

From what preceeds we are allowed to consider only a half-arch, leant against a

FIG. 3.



fixed surface at A, and solicited by a horizontal force at C.

5. (Figure 3.) Let ab, a_1b_1, a_2b_2, be the joints of an arch; P_1, P_2, the vertical directions of the weights of the parts ab b_1a_1; abb_2 a_2; including the loads on the

parts bb_1, bb_2,......; P_1, P_2 acting through their common centres of gravity.

The horizontal force Q, at the crown, combined with the reaction R_2 at the joint a_2b_2 holds the part abb_2a_2 in equilibrium and similarly for the reactions on other joints.

At the points where the direction of Q cuts P_1, P_2, combine those forces with Q as shown in the figure; the resultant of Q and P_1 cuts joints a_1b_1 at A_1, which is therefore the *centre of pressure* on that joint. As the weight abb_2a_2 with its load equals P_2 and is the weight on joint a_2 b_2, the resultant of P_2 and Q will give the force acting on a_2b_2 in direction, position and magnitude; it cuts a_2b_2 at A_2, which is therefore the *centre of pressure* on that joint.

In the same way the resultants and centres of pressure on all the joints may be determined. A broken line connecting these *centres of pressure* on the various joints will be called *the line of the centres of* pressure or more briefly, *the line of resistance.*

For voussoirs indefinitely small, it approaches indefinately a curved line and will be called *the curve of the centres of pressure* or *curve of resistance.*

That granted, in order that the arch may remain in equilibrium, it is necessary:

(1.) That the points of intersection C', A_1, A_2, fall in the interior of the respective joints ab, a_1b_1, a_2b_2. If for any joint this is not so; *e. g.*, if the point A_1 was above b_1, the mass $abb_1\ a_1$ would then turn around the edge b_1, as an unresisted couple would be formed. To explain: suppose the resultant R_1 to pass outside of joint a_1b_1; conceive two equal opposed forces, each equal to R_1 to act at edge b_1; this does not disturb the equilibrium; then R_1 the force acting through A_1 (which is outside the joint) with its equal but not directly opposed force at b_1, would form the unresisted couple in question which causes overturning.

(2.) That the directions c_1A_1, c_2A_2 of the pressures upon the joints do not make angles with the normals to the respective joints which exceed the angle of friction.

If it was not so, sliding at the joints in question would occur of the mass above or below.

However, the friction of the materials usually employed in construction is sufficiently great to not give cause for fear as regards sliding, generally.

It is very easy to alter the direction of the joints should sliding be apprehended, hence it will not be considered further.

The two former requirements refer to *stability*, supposing no crushing of the material occurs. The last requirement pertains to the *strength* of the masonry at a mortar joint.

(3.) The resultant of the pressure on any joint must not pass so near the edge that crushing of the mortar or the stone (or brick) may occur. Of course in practice a certain factor of safety would be used, but as this subject will be treated in full later, we shall only remark here that reasons will be given further on why the true line of resistance should not depart at any joint more than one-sixth the depth of joint from its centre, or in other words,

should everywhere be confined to the middle third of the arch ring and in fact it is best to use still narrower limits.

It is to be remarked that the foregoing theory does not require horizontal resistance in the spandrel, which is not generally built with the same care that is taken in the construction of the arch stones, and thus cannot generally be regarded as unyielding; hence when a *line of resistance* such as $C'A_1 A_2$ passes, somewhere, out of the arch ring, a serious derangement of the arch may occur, even though the spandrel may prevent its falling: hence it appears to be a poor construction to build such an arch in preference to an arch in which the resultant pressures on the joints everywhere keep within the limits prescribed. This will be adverted to again.

If no line of resistance can be drawn within the prescribed limits, or crushing is feared on any joint, the depth of the voussoirs must be increased on part or the whole of the arch; or the profile may be altered; or, finally, we may combine both of these methods to secure stability.

6. We shall now enter into detail to show how lines of resistance can be drawn through various points of the arch ring.

Let us consider as in Art. 2 a slice of the arch 1 unit thick. In Fig. 4, let Q=the horizontal thrust at the point M of the

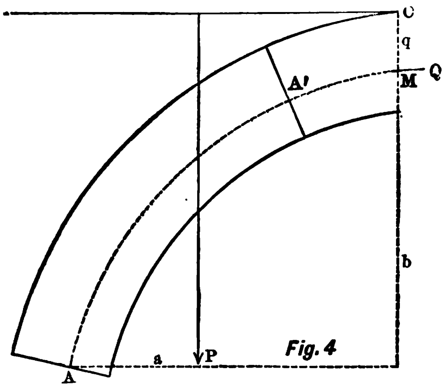


Fig. 4

crown joint (compare Fig. 3 throughout); q, its vertical distance below the apex C; P=weight of arch and load C A on joint at A; a=horizontal distance from A, the centre of pressure on the joint, to the vertical passing through the centre of gravity of P; h=vertical distance between C and

A; b = vertical distance between M, the point of application of Q at the crown, and A.

If we consider another point A′ of the curve of resistance, at a vertical distance above A = e, we shall have an analogous notation P′, a', b'.

If we know the points M and A, we have, taking moments about A,

$$a\,P = b\,Q \quad \therefore\ Q = \frac{a\,P}{b} \quad \text{............(1).}$$

If we know any two points as A, A′, $a\,P = b\,Q = (b' + e)\,Q$; also $a'\,P' = b'\,Q$,

hence,
$$Q = \frac{a\,P - a'\,P'}{e} \quad \text{............(2).}$$

$$q = h - \frac{a\,P}{Q} \quad \text{..............(3).}$$

Having obtained from eq. (1) or eqs. (2) and (3), Q and its point of application at the crown, we find where the resultant pressures cut each joint as in Art. 3.

If the first curve drawn passes outside of the prescribed limits in one or more places, take points on the limiting curves

opposite the points of maximum departure, and by eqs. (2) and (3), pass a curve through two of these points.

If the arch is stable at all, it will almost invariably be found in practice that the last curve so drawn will fulfill the required conditions of remaining in the prescribed limits. If not a third approximation may be tried, and so on.

This is Dr. Scheffler's method of drawing a line of resistance within prescribed limits and in practice, after becoming familiar with the leading cases, the first trial is generally sufficient.

7. Let us proceed to show how to find the centres of gravity of the weights abb_1a_1, abb_2a_2 (Fig. 3), as also the magnitudes of those weights. If the arch is loaded with any weights, reduce them to the same specific gravity as that of the masonry of the bridge supposed homogeneous, as follows: Lay down these weights in their exact positions on the arch, and alter the vertical dimensions to conform to the specific gravity of the stone. We shall thus substitute blocks of stone, by scale,

for the surcharge of earth, water, etc., or the rolling load.

We now divide the horizontal through A (Fig. 5) into an appropriate number of parts and through these points of division

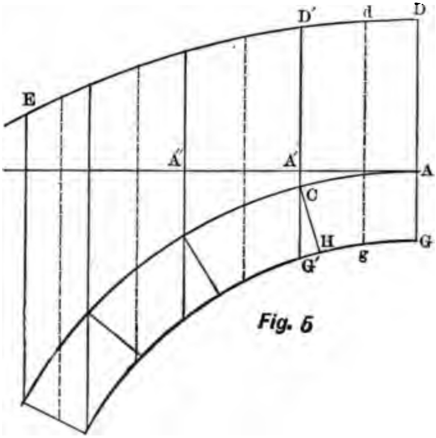


Fig. 5

draw verticals from the intrados to the curve DE that limits above the reduced load.

Regard each trapezoid DGG′D′ as a rectangle, and calculate its surface by multiplying its horizontal width AA′ by

the mean vertical *dg*. Next regarding the centre of gravity of each trapezoid as that of the corresponding rectangle, we shall find the centre of gravity of the trapezoid DD′G′G, for example, to be upon the mean vertical *dg*, which equally divides the horizontal AA′. Draw through C, the joint CH; the weight DD′G′G will be considered as resting on the joint CH, which is in excess by the small triangle CG′H, an error too small to be regarded in flat arches.

Scheffler gives a graphical construction for correcting the joints to correspond better to the weights, but it is very tedious in practice and defaces the drawing with too many many lines, so it is not given. In article 11 will be found a method that will insure all desirable accuracy for *any* form of arch.

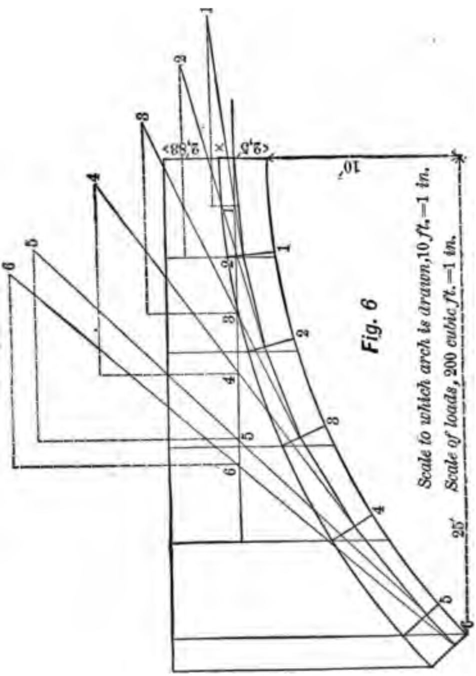
The assumption that the whole weight vertically over a voussoir acts on it is not strictly true, whether the spandrel is of earth or masonry; for part of it is doubtless distributed in a different manner by aid of the friction of the particles, or in some cases from the spandrel itself acting as a sort of arch. Again, any horizontal spreading of the haunches of the arch is

partially hindered by horizontal resistances in the spandrels. These influences though are on the side of safety, particularly when the dynamic effects of moving loads are considered. In the case of culverts the earth exerts a thrust, as to which and the treatment of culverts and tunnel arches, see Science Series No. 42.

Assuming the loads over the arch to act as above, we multiply the surface of each trapezoid by the horizontal distance of its centre of gravity from A; the sum of these moments divided by the sum of the trapezoid surfaces (which are also the volumes), will give the horizontal distance from A to the centre of gravity of the whole part considered. This method will thus give us the weights P_1, P_2, - - - (Fig. 3), as well as the horizontal distance of their centres of gravity from the crown.

8. *First Example.*—Let us illustrate by an example of a railroad bridge (Fig. 6) of 50 ft. span and 10 ft. rise, the arch being a segment of a circle; voussoirs 2.5 ft. deep; the spandrel walls, of the same specific gravity as the voussoirs, rising 2.83 ft.

above the summit of the arch ring. The



arch is 7 ft. thick, but we shall consider but a vertical slice of it 1 ft. thick.

In the following table the first column gives the number of the joint from the crown; the second (w) the width of the horizontal divisions AA′, A′A″- - of Fig. 5; the third (v) the corresponding mean heights *dg* - - -; the fourth (s), the product of these dimensions, giving thus the surface of each trapezoid. Column (c) gives the distance of the centre of gravity of each trapezoid from the crown; column (m) the product of (s) and (c). Now we cumulate, going from the crown, in the next two columns, these surfaces (*s*) and products (s $\times$ c); column (S) being formed by adding the surface of each trapezoid to the total surface, just found, which precedes it. The last quantity in column (S) should = sum of colum (s).

In the same way column (M) contains the continued sum of column (m), and hence its last number should equal the sum of column (m). Dividing now the numbers in column (M) by the corresponding ones in column (S) we get, column (C), the horizontal distances of the centre of gravity of each weight P_1, P_2—

—, corresponding to joints 1, 2 - - -, from the crown.

Joint	w	v	s	c	m	S	M	C
1	5	5.4	27.	2.5	67.50	27.	67.50	2.5
2	5	6.1	30.5	7.5	228.75	57.5	296.25	5.1
3	5	7 6	38.	12.5	475.	95.5	771.25	8 1
4	5	9.8	49.	17.5	857.50	144.5	1628.75	11.3
5	5	13.2	66.	22.5	1485.	210.5	3113.75	14.7
6	1.75	14.5	25.4	25.9	657.86	235.9	3771.61	16.
			235.9		3771.61			

The preceding table shows that the *surface* (or volume, for a slice 1 ft. thick) of the half arch with its load equals 235.9 sq. ft.; its moment as to the crown is 3771.61 and the distance of its centre of gravity from the joint at the crown is 16 ft. Let it be required to pass a curve of resistance through the crown joint, ⅓ of its depth from the summit of the arch and through joint 6 at ⅓ of its depth above its lowest point. By measurement on the drawing (Fig. 9) we find $a=25.6-16=9.6$, $b=11.2$. We have also $P=235.9$ cubic feet of stone; hence by formula (1), Art. 6,

$$Q=\frac{aP}{b}=\frac{235.9\times 9\,6}{11.2}=202 \text{ cubic ft. stone}$$

which may be reduced to tons, when desired, by multiplying by the weight in tons of a cubic foot of stone.

If now at the points of intersection of the horizontal through the point of application of Q at the crown, with the verticals passing through the centres of gravity of the surfaces given in column (S), [P_1, P_2, - - -, of Fig. 3], we combine these weights with Q, the points of intersection of the resultants of Q with these weights P_1, P_2,......,with the corresponding joints, will be points in the curve of resistance sought.

For example, to determine where the line of resistance cuts joint 4, lay off the distance in column (C), 11.3 horizontally from the crown, then on a vertical lay off upward from this point the corresponding weight on joint 4, given in column (S) 144.5; drawing a horizontal line through the last point found=Q=202, we get the resultant by completing the triangle of forces.

Producing this resultant to intersection with joint 4, will give the centre of pres-

sure on that joint. It will be advisable, in practice, to prick off the centres of gravity, taken from column (C), at one operation and number each one with the number of the corresponding joint to avoid mistake.

On continuing this construction for each joint, we shall find that the line of resistance remains within the inner third of the arch ring.

It may be remarked that the small triangle mentioned is in excess only for the joint in question; thus this error is not carried on.

The ordinary method of constructing a line of pressures is to combine any resultant with the next weight following, regarded as concentrated at its centre of gravity. By this construction any small error in draughting is carried on, whereas, by the former method, it is confined only to the joint where it occurs first.

With accurate instruments and care, using a sufficiently large scale, this method should answer all the requirements of accuracy, and will generally be found the

shortest in the end; whereas, with many joints, it is difficult to locate this curve precisely by the ordinary method.

We have made use in the above example, of formula (1) to compute the thrust at the crown. This can preferably be found without computation, as follows: at the point 6 on the horizontal through the crown, lay off, to the scale of force, vertically upwards the weight of half arch and load = 235.9 and through the extremity of the line, draw an indefinite horizontal. The intersection of the latter with the line drawn through the point 6 first mentioned and the assumed centre of pressure on joint 6, will cut off a distance on this horizontal, equal to Q to the scale of force. This value of Q is then to be laid off in constructing all the other triangles of force. This method can likewise be followed in the next example if preferred.

9. *Second Example.*—(Fig. 7.) Suppose a load of two 40-ton engines, one on each side of the crown, over divisions 2, 3, and 4, i. e., 15 ft. along the rails. We shall suppose it to bear only on 6 ft. of the thickness of the viaduct. Calling the weight of a cubic foot of stone = .07 ton and h, the height of the block of stone 15 ft. long by 6 ft. wide that is required to weigh as much as one engine; we have

$$6 \times 15 \times h .07 = 40 \quad \therefore h = 6.3$$

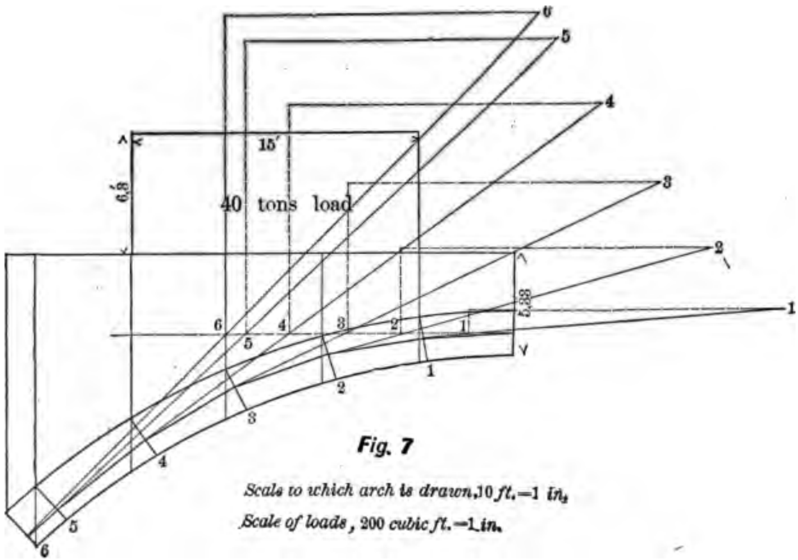


Fig. 7

Scale to which arch is drawn, 10 ft. = 1 in.

Scale of loads, 200 cubic ft. = 1 in.

We now form the following table which refers to Fig. 7, which, as the arch and load is symmetrical, represents, as before, only one-half the arch.

Joint	w	v	s	c	m	S	M	C
1	5	5.4	27	2.5	67	27	67	2.5
2	5	12.4	62	7.5	465	89	532	6.
3	5	14.	70	12.5	869	159	1401	8.8
4	5	16.	80	17.5	1408	239	2809	11.8
5	5	13.2	66	22.5	1485	305	4294	14.1
6	1.75	14.5	25	25.9	658	340	4952	15.
			330		4952			

A line of resistance passing through the middle of the crown, the point on the springing joint, as before, will be found to be contained inside of the limiting curves, and is drawn as in Fig. 7, taking care to lay off the centres of gravity on the prolongation of Q. We find in this case $a = 25.6 - 15 = 10.6$, $P = 330$, $b = 10.7$.

$$\therefore Q = \frac{330 \times 10.6}{10.7} = 327 = [illegible] = 23 \text{ tons.}$$

If it is desired to draw the curve corresponding to the minimum of the thrust in the limits chosen (see Art. 19.) we resort to equations (2) and (3). As the nearest approach of the last line of pressures drawn to the outside limiting curve, is at joint 2; pass a curve of resistance through the point of intersection of that outside limiting curve with the second joint and the previous point at the springing joint.

We find $P = 330$, $a = 10.6$, $e = 9.8$ and from table 2, column (S) $P_1 = 89$; from column (c) and the drawing $a_1 = 9.8 - 6 = 3.8$.

From (2), Art. 6,

$$\therefore Q = \frac{a\,P - a_1\,P_1}{e} = \frac{3498 - 338}{9.8} = 322$$

From (3)

$$q = h - \frac{a\,P}{Q} = 11.93 - \frac{3498}{} = 1.04$$

Laying off this latter distance. from the summit of the arch ring. downwards, we draw the curve as before. It is everywhere within the proper limits, and of course touches the upper middle third limit at joint 2 and the lower middle third limit at joint 6, as assumed.

If we suppose a weight of 13.3 tons to rest on division 3 on both sides of the crown, along 5 ft. of the length of the rails, we shall find by forming a table and constructing the line of resistance as in the last case above, that it passes slightly below the upper limit at the crown, and is everywhere contained in the middle third of the arch ring.

A curve of resistance for a uniform load of 1.5 tons per foot along the whole length of the bridge can be drawn to follow very closely the curve drawn in the first example.

One or two more suppositions of isolated weights, symmetrically placed. were made, but in all cases it was found that a curve of resistance could easily be drawn in the inner third of the arch ring. The thrust is too small to fear crushing, and the directions of the thrust are inclined to the normals of the arch joints at angles much smaller than the angles of friction, hence sliding is not to be feared.

The curves of resistance drawn in the preceding examples are not necessarily the true ones, otherwise we should at once conclude that thus far the arch had stability. The true curve depends upon the elastic yielding of the arch to the weights acting on it and we shall see later how the aid of a few principles from the theory of elasticity will enable us to locate it approximately. For the present, we shall continue the subject by showing how

any trial line of resistance can be made to pass through any three points of an arch ring, either unsymmetrical or unsymmetrically loaded. The method first given below requires the solution of some equations: subsequently in Art. 15 a purely graphical method is developed which will doubtless generally be preferred.

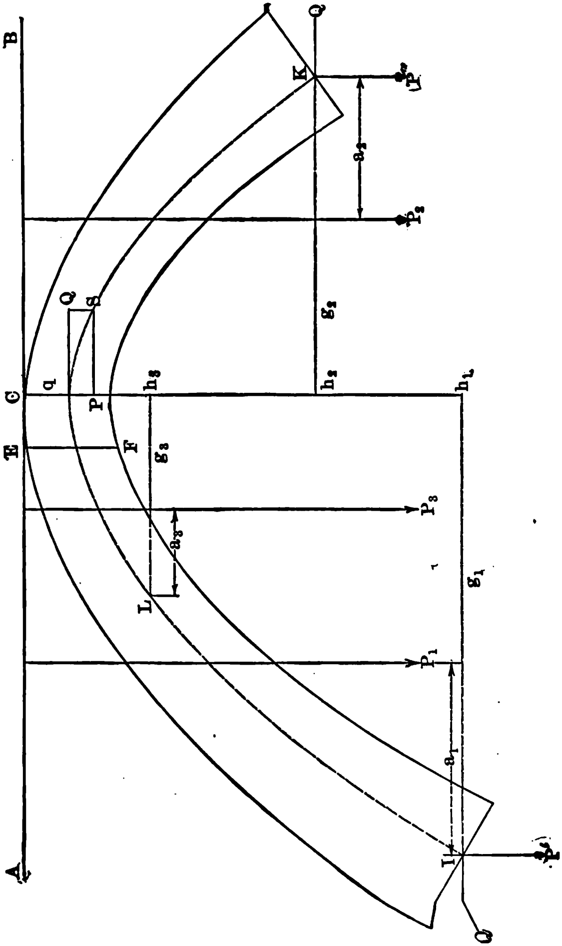
Unsymmetrical Arches.

10. In unsymmetrical arches, or arches unsymmetrically loaded, there is a joint, EF (Fig. 8), generally near the crown, at which the thrust is horizontal. Where the arch is only solicited by vertical forces, by compounding them with this thrust, as before, we find the resultants on every joint, and it is evident in this case that the horizontal thrust is the same all through the arch.

It is more convenient however, to find the inclined thrust at the crown and combine the partial weights with it, to find the resultant on each joint.

Problem: To find this inclined thrust and its point of application:

FIG. 8.



Let I, L, K be three points through which to pass a curve of resistance (Fig. (8), $\overline{ACB}$ a horizontal line drawn through the highest point of the extrados, and let there be:

Q, the horizontal thrust, *i. e.* the horizontal component of any one whatsoever of the pressures acting through I, L, K.;

P, the vertical component of the pressure S at the crown joint, which will be considered positive, if it is directed upwards, as regards pressure from the right part upon the left.

Let q, be the vertical distance of the point of application of S below the horizontal $\overline{ACB}$; g_1, h_1; g_2, h_2; g_3, h_3, the horizontal and vertical co-ordinates of I, K and L as to the point C as the origin;

P^{I}, P^{II}, P^{III}, the vertical components of the pressures acting through I, K and L; P_1, P_2, P_3, the weights of the segments C I, CK, CL, with their loads;

a_1, a_2, a_3, the horizontal distances of the centres of gravity of these segments CI, CK, CL, from the points I, K and L respectively.

To abbreviate, let us put:

$h_1 - q = b_1 \qquad h_2 - q = b_2 \qquad h_3 - q = b_3$

$g_2 + g_3 = d_1 \qquad g_1 - g_3 = d_2 \qquad g_1 + g_2 = d_3$

$h_2 - h_3 = e_1 \qquad h_1 - h_3 = e_2 \qquad h_1 - h_2 = e_3$

Observe that arch CI is in equilibrium under the action of the left reaction (P^1 acting up and Q acting to the right being its components) the thrust S at the crown (P acting up and Q acting to left being its components) and the weight of arch CI and load P_1 acting downwards.

Similarly, the part CK is in equilibrium under the action of P'' (acting up) and Q (acting to left) at K, the force P (acting down) and Q (acting to the right) at the crown and the weight P_2.

Lastly, the part CL can be dissociated from the rest and conceived to be in equilibrium under the action of the reaction at L acting upwards, the forces P (acting up) and Q (acting to the left) at the crown and the weight P_3 of CL and load.

Balancing vertical components for the parts CI and CK respectively, we have,

$$P' + P = P_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$P'' - P = P_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Next taking moments about I, K and L of the forces holding in equilibrium the part CI, CK and CL respectively.

$a_1\ P_1 - g_1\ P = b_1\ Q$ .... (3)

$a_2\ P_2 + g_2\ P = b_2\ Q$ .... (4)

$a_3\ P_3 - g_3\ P = b_3\ Q$ .... (5)

If the third given point L of the curve of pressures is upon the joint at the crown C, the value of q is known, and we have: $g_3 = o$, $h_3 = q$, $P_3 = O$. From eqs. (3) and (4) we find

$$P = \frac{a^1 b_2 P_1 - a_2 b_1 P_2}{g_1 b_2 + b_1 g_2} = \frac{a_1 e_1 P_1 - a_2 e_2 P_2}{e\ {}_2 d_3 - e_3 d_2} \text{ .... } (6)$$

$$Q = \frac{a_1 P_1 - g_1 P}{b_1} = \frac{a_1 d_1 P_1 + a_2 d_2 P_2}{e_2 d_3 - e_3 d_2} \text{ .... } (7)$$

If L is not upon the joint at the summit, we find *

$$P = \frac{a_1 e_1 P_1 - a_2 e_2 P_2 + a_3 e_3 P_3}{e_2 d_3 - e_3 d_2} \text{ .... } (8)$$

$$Q = \frac{a_1 d_1 P_1 + a_2\ a_2\ P_2 - a_3 d_3 P_3}{e_2\ d_3 - e_3 d_2} \text{ .... } (9).$$

$$q = h_1 - \frac{a_1 P_1 - g_1 P}{Q} \text{ .... } (10)$$

---

* Add-(4) and (3) and call the sum eq. (11); also subtract (5) from (3.) Place the values of Q equal to each other in this last eq. and eq. (11); reducing, bearing in mind that $d_2 - d_3 = d_1$, $e_2 - e_3 = e_1$, &c., we find P as in eq. (8). Substitute this value of P just found in eq. (11) and deduce Q. which gives eq. (9). Eqs. (6) and (5) are only particular cases of eqs. (8), (9) and (10) when $P_3 = O$.

Example 1. Fig. 9 represents the same viaduct, before considered in Art. 8, with a load of 40 tons on 15 feet of length over divisions 2, 3 and 4, on one side of the arch only. The table of Art. 8 refers to the right half of the arch: the table of article 9 to the left side.

Let us pass a curve of resistance through the middle of the crown and through a point on each springing joint, ⅓ depth joint above its lower edge.

We find from the drawing and tables.

$$g_1 = g_2 = 25.6, \qquad b_1 = b_2 = 10.75$$

$$P_1 = 330, \qquad p_1 = 15, \qquad a_1 = 10.6$$

$$P_2 = 236, \qquad p_2 = 16, \qquad a_2 = 9.6$$

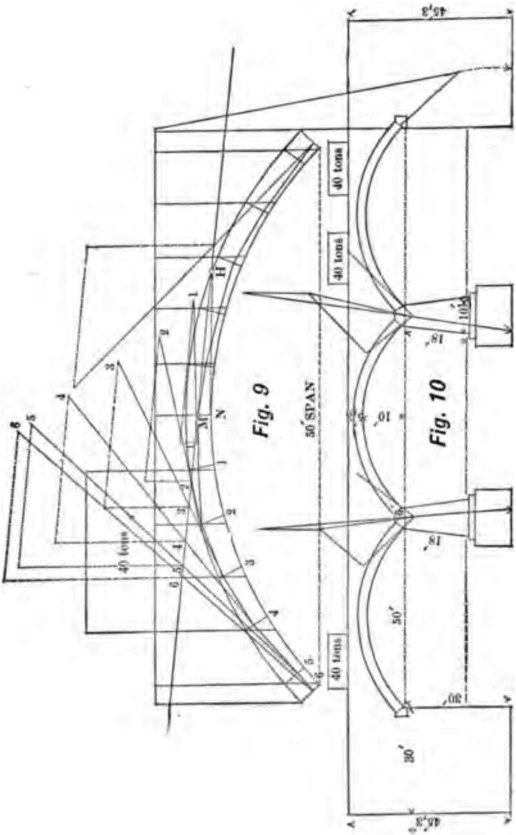
From (6):

$$P = \frac{a_1 b_2 P_1 - a_2 b_1 P_2}{g_1 b_2 + b_1 g_2} = 24 \text{ cubic ft. of stone;}$$

from (7):

$$Q = \frac{a_1 P_1 - g_1 P}{b_1} = 268 \text{ cubic ft. of stone;}$$

From M, the middle of the crown joint, lay off downwards $\overline{MN} = P$, also $\overline{NH} = Q$ on the horizontal through N; $\overline{MH}$ will then represent the resultant on the crown joint


40 tons
40 tons
40 tons
40 tons
50' SPAN
Fig. 9
Fig. 10
H
N
M
50'
30'
18'
18'
10'
45'.3
45'.3

in direction, position and magnitude; and by combining it with the weight of each artificial voussoir and load, on each side of the crown, each acting through its centre of gravity, we evidently obtain the resultants on the various joints in direction, position and magnitude, and therefore can trace the curve of pressures. For example, to find the resultant on the third joint on the left side of the arch: draw a horizontal line through M and lay off on it the distance of the centre of gravity of the three first divisions, from M, which by Table 2 (Art.8), column C, is found to be 8.8.

Draw a vertical through this point and from its point of intersection with $\overline{MH}$, lay off upwards the weight 159 (column S) of the three divisions in question.

From the upper extremity of this last line draw a line ∥ and equal to $\overline{MH}$; completing the parallelogram of forces as per figure, the point where the resultant cuts joint 3 is the centre of pressure of that joint, and the resultant is given in magnitude, position and direction by the diagonal.

The construction for the other joints is the same.

The nearest approach of the curve of pressures to the extrados is on joint 2, of the left side of the arch, where it is only three-tenths (.3) of a foot (on a large scale drawing it was found to be .35) from the edge. The nearest approach to the intrados is at joints 3, 4 and 5 on the right, being only about .7 to .75 from the edges at those joints.

Example 2 Draw a line of resistance for the bridge, loaded as above, through the lower middle third limit at the left springing, the upper middle third limit at the right springing and at a point on the crown joint 1.1 ft. above the intrados. It passes above the middle third limit at joint 2 on the loaded side, its maximum departure, and just touches the lower limit at joints 1 and 2 on the unloaded side.

Example 3. By aid of formulas (8), (9) and (10), draw a line of resistance through the lower middle third limit at the left springing and the upper middle third limit at joint 2 under the load and also through the upper limit at the the right springing joint.

The thrust at the crown will be found now to act 0.76 ft. below the centre of the joint, its horizontal component being 294 cu. ft. and its vertical component 18 cu. ft. The line of resistance everywhere keeps within the middle third limits except at joints 1 and 2 on the right where it passes 0.14 and 0.12 respectively below the limit.

Example 4. A load of 13-3 tons was assumed on division, 3 on one side, and it was found that a curve of pressures

could be drawn, for this eccentric load, within the inner third of the arch ring.

If the backing is raised higher, thus making the bridge weigh more, a rolling load will have less effect upon it; hence a less depth of keystone may be used. Other things the same, it is a simple question of economy, considering the approaches, whether to increase the height of surcharge above the arch ring, or the depth of the arch stones.

Fig. 10 shows the effect of rolling loads in different positions, on the piers; the middle bay not being loaded but with its own weight, the end spans as per figure. The resultants at the springing joints we have before determined; combining the two on any pier with the weight of pier, according to the usual rule for three forces not intersecting in one point, we obtain the final resultants on the bases of the piers.

It is seen from the figure that the 40 tons on both sides produces a more hurtful effect on the pier than a 40 ton load on one side only.

By combining the weight of abutment with the thrust on it, we find that the centre of pressure on the foundation course is sufficiently within the limits for most cases in practice.

The dotted line in the abutment gives

the centres of pressure of all the forces acting on each joint for the joints in question. For example, to find where this centre of pressure is on the springing line, produced, we combine the inclined resultant on the arch joint at the springing with the weight of the abutment above the springing line, acting through its centre of gravity. This resultant makes an angle with the vertical of only 25°, hence sliding on the springing course is not to be feared, if the abutment is solidly built.

The lines of resistance as drawn for the three arches, piers and abutments are not necessarily the true ones.

Further on will be given a method of locating approximately, the true line of resistance for a well built arch, with thin mortar joints, between immovable abutments. The abutments in the figure are of such proportions as to be practically immovable, as the centre of pressure on the base is near its centre, but not so the piers The first pier on the right tends to lean to the left, the second one to the right. This tendency is resisted too by the

central arch, which thus puts forth a stronger horizontal thrust than assumed, corresponding to a line of resistance passing nearer the intrados at the crown and the extrados at the skewbacks, the maximum efforts being produced when the line of resistance passes very near these curves so that no crushing ensues. We are not able to locate this line with our present knowledge, but it is plain that this central arch will put forth its maximum effort if necessary, to prevent much motion inwards of the tops of the piers, so that the centres of pressure on their bases will not depart as far from their centres as the figure shows. As there will be some motion however, it tends to cause the line of resistance of the other arches to travel down the skewbacks at the piers and to move up the crown joints, from the slight increase of span, thus giving rise to a less horizontal thrust from those arches, which again tends to correct the eccentricity of the thrust on the piers. If preferred, the lines of resistance can be redrawn in these arches, corresponding to a minimum thrust

(within reasonable limits) of the outer arches and a maximum thrust of the central arch, when the stability of the piers will be more apparent. Experience indicates that piers of the proportions shown are perfectly stable.

CHAPTER II.

11. A More Correct Method of Making out the Table of Weights and Centres of Gravity.

The method of finding the weights and centres of gravity given in Art. 7, although sufficiently correct for flat arches with a small depth of key, is not so for thick arches approaching the semicircular or elliptical in form. The following is suggested by the author as giving all desirable accuracy with but little more labor than Scheffler's method.

The arch is preferably divided into a number of *equal* voussoirs (see Fig. 11), and the vertical lines drawn from the upper ends of the joints to the reduced contour of the surcharge, divides the latter into trapezoids. As before, we draw the medial dotted lines, which will be assumed to pass through the centres of gravity of the trapezoids, though the latter can be found exactly by the usual graphical con-

struction if desired. The area of a trapezoid = width × mean height = w v. On multiplying the area by the distance from the crown to the medial line of the trapezoid (c) we have the moment m = (w v) c, about the crown, for any trapezoid.

The quantities *w*, *v*, *s*, *c* and *m* for Fig. 11 are entered in the table below, being the upper numbers corresponding to the joint given in the first vertical column. The corresponding quantities for the voussoirs are the lowest numbers of the horizontal rows.

Calling r, in Fig. 11, the radius of the extrados, r_1, that of the intrados and n the proportion of the circumference included by the voussoir, we have its content $=\dfrac{\pi(r^2-r_1^2)}{n}$ for a thickness of 1. Now this is equal to $\left[\text{the depth } (r-r_1) \times \text{the } \textit{middle length} \left(\dfrac{1}{n} 2\,\pi\,\dfrac{r+r_1}{2}\right)\right]$ hence measure the middle length and depth on a drawing, their product will give the required volume of a voussoir ($= 2.35 \times 2 = 4.7$ in this case).

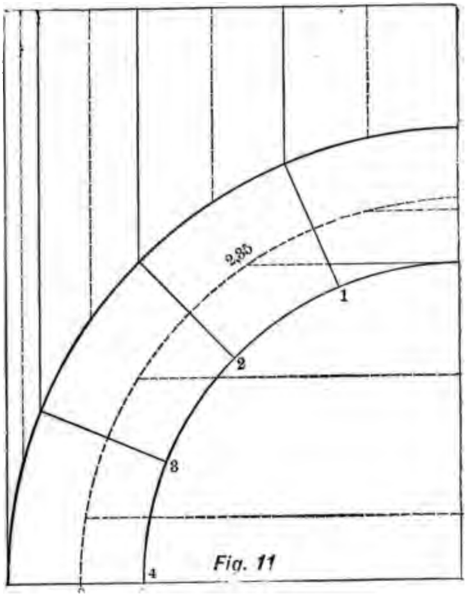


Fig. 11

	w	v	s	e	m	S	M	C
1	2.72 2.35	2.13 2.	5.79 4.7	1.38 1.18	7.99 5.55	10.49	13.54	1.29
2	2.27 2.35	3.16 2.	7.17 4.7	3.88 3.36	27.82 15.79	22.36	57.15	2.55
3	1.51 2.35	5. 2.	7.55 4.7	5.77 5.03	43.56 23.64	34.61	124.35	3.59
4	.05 2.35	7.2 2.	3.6 4.7	6.77 5.92	24.37 27.82	42.91	176.54	4.11
			42.91		176.54			

The centres of gravity of the voussoirs will be assumed to lie on the (dotted) centre line of the arch ring and midway between the joints; the distances from these points to the vertical through the crown give the arms in column (c). The volume (4.7) of a voussoir multiplied by its *c*, gives the corresponding *m* of the table.

This manner of considering the voussoirs and surcharge separately is continued, until in columns S and M the quantities referring to the same joint are combined by the continued addition of the quantities in colums (s) and (m) respectively.

If the voussoirs are taken the same size, there is really no necessity of entering their dimensions; simply giving their common area in column (s).

When the voussoirs are taken small enough this method gives all desirable accuracy.

Concentrated loads on the arch are easily included by introducing a third row of numbers, for any voussoir affected, just above those given for the trapezoids as will be fully explained in a subsequent article.

12. A Convenient Method of Drawing a Trial Line of Resistance in an Arch.

Let fig. 12 represent a semi-circular arch of 100 ft. span, 3 ft. key and 3 ft. depth of surcharge over the crown of the same specific gravity as the voussoir. The live load extends from the crown to the right abutment and weighs 3,000 pounds per foot of rails. If this bears on a width of 6 feet it is equivalent to a layer of stone of the same density as the voussoir (150 lbs. pr. cu. ft.) 3. 4 ft. high, as shown in the figure.

The spandrel was divided up by vertical lines, 5 ft. apart for 40 ft. from the crown, then 2 ft. apart for the next 10 ft., and 1 ft. apart for the remaining 3 feet. The joints 1, 2, 3, . . . are then drawn as in the figure.

The following is a condensed table of loads (S in cubic feet) and distances from the crown to their centres of gravity (C).

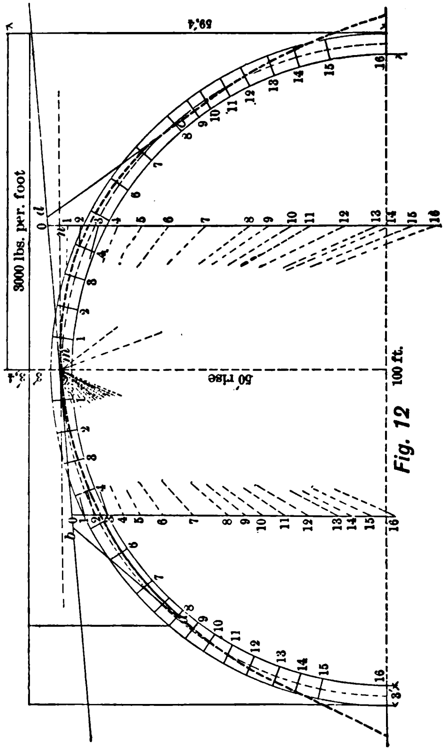


Fig. 12

LEFT HALF.

Joints	1	2	3	4	5	6	7	8
S	30	62	100	146	201	272	362	475
C	2.5	5.1	7.8	10.8	14.	17.4	21.1	25.

Joints	9	10	11	12	13	14	15	16
S	529	589	656	732	819	869	925	1004
C	26.6	28.3	30.	31.7	33.5	34.5	35.5	36.8

RIGHT HALF.

Joints	1	2	3	4	5	6	7	8
S	47	96	151	214	286	374	481	611
C	2.5	5.	7.7	10.5	13.6	16.8	20.2	23.9

Joints	9	10	11	12	13	14	15	16
S	672	739	812	895	989	1042	1101	1184
C	25.4	27.	28.6	30.3	32.	33.	33.9	35.2

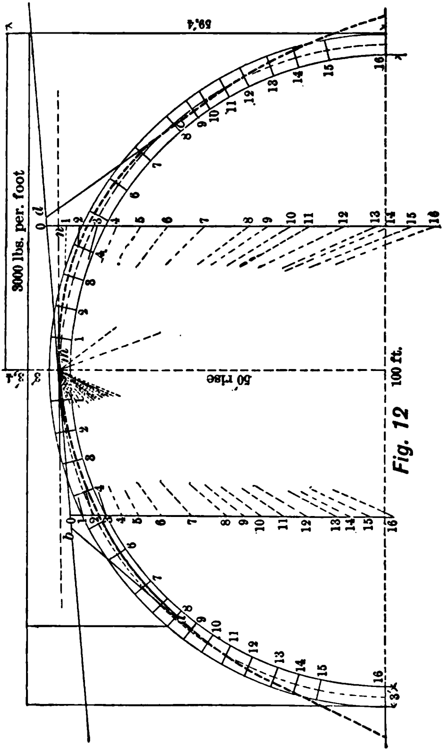


Fig. 12

Let us pass a line of resistance one-twelfth the depth of arch ring below the centres of joints 8 and the crown joint, or through *a*, *c* and *m*. By the formula method of art. 10 we find by aid of a drawing, &c., P = -- 32.8, Q = 444.6. (It will be instructive for the reader to test these values by the purely graphical method of art. 15, which is generally to be preferred).

From *m* lay off to the right horizontally, Q = 444.6 = *mn*; then vertically upwards, P = 32.8 = *no*: *om* represents the resultant at the crown joint.

Lay off on $\overline{om}$ produced $\overline{mo}$ to the left and equal to $\overline{om}$. Through the points o thus determined, draw verticals and lay off from o the distances $\overline{01}$, $\overline{02}$, - - - , equal to the values of S pertaining to joints 1, 2, - - - , as taken from the tables pertaining to the right and left sides respectively. Straight lines from *m* to 1, 2, - - - , represent the resultants of the thrust at crown and load down to joints 1, 2, - - - , in magnitude. Their positions are found as follows: draw a horizontal through *m*, and

lay off on it the numbers in column C; the first table referring to the left half of the arch, the last table to the right half.

From the points so found draw vertical lines to intersection with $\overline{mo}$, produced if necessary, which thus give the points where the inclined thrust at m is to be combined with the weight from the crown to any joint, to find the resultant on that joint; whose intersection with it is thus the centre of pressure for that joint.

Thus the weight from the crown to joint 8 on the left, acts 25′ to left of m; lay off 25′ on the horizontal through m, then drop a vertical to intersection b with mo; then draw $\overline{ba} \parallel m8$ of force diagram for left of arch, to find a the center of pressure for joint 8. Similarly d and e are found for joint 8 on the right. These should be the first constructions made to test the values of P and Q found, which correspond to the line of resistance passing through a, m and e.

The line of resistance thus drawn passes below the middle third of the arch ring

on the unloaded side, the following amounts in feet: at joints 2, 3, 4, 5 and 6, .3, .4, .3, .2 and .1 respectively; it then crosses the arch ring, passes above the middle third about joint 12, and cuts the springing joint 4.5 feet outside of the arch ring.

On the loaded side it passes above the middle third 0.1 at joints 4 and 5; then crosses the centre line and is just tangent to the lower middle third limit at joint 10, below which it again crosses the arch ring and passes into the abutment, cutting joint 16 about 3 feet outside of the arch ring.

Exercise. Draw a line of resistance for the part *a m c* regarded as a segmental arch, through the upper middle third limit at joint 8 on the left, the lower limit at joint 8 on the right and 1.25 ft. above the intrados at the crown.

We should find P=—23.8, Q=449.1 and the line of resistance everywhere keeps within the middle third, barely touching the lower limit at joint 2 on the left and passing 0.16 ft. inside of the upper limit at

joint 3 on the right and corresponding nearly to the maximum and minimum of the thrust in the limits of the middle third (see Art. 20). The span of the arch *a m c* is 75.45 feet, the rise 17.2 ft. or between ¼ and 1/5 of the span.

If for a moment we regard the *unaided* semi-circular arch first considered; since the line of resistance (or in fact any line of resistance that can be drawn inside the arch ring of the upper portion) passes outside the arch ring at the abutments, the arch will fall, the parts 8–16 rotating outwards about joints 16 and the crown descending. But with spandrels built of solid masonry up to about joints 8 (called the joints of rupture), the parts 8–16 can be regarded as almost immovable and the part *a m c* can be approximately treated as a segimental arch on fixed abutments.

However, as the higher the abutments the more their tops will yield to a horizontal thrust, the depth of arch ring determined for the segimental arch *a m c*, regarded as resting on immovable abutments, should be slightly increased to

allow for the slight horizontal spreading at *a* and *c*. This spreading is due partly to the elastic yielding of the abutment from 8 down to the foundation and partly to the closing up the joints of the rather fresh mortar in the vertical joints of the spandrels when the centres are struck.

Some constructors, especially the French engineers, increase the depth of arch ring from the crown to the abutments so that the true line of resistance shall not leave the middle third (or other limits) anywhere. This is, of course, the best way to build a semi-circular or elliptical arch, the abutments being built with joints inclined (about at right angles to the thrust) and in fact treated as a part of the arch in finding the true line of resistance.

13. The Usual Method of Drawing a Line of Resistance. Equilibrium Polygon.

In Fig. 13, representing half an arch, suppose the thrust S at the crown is known in position, direction and magnitude and that the weights P_1, P_2 and P_3 of the suc-

cessive voussoirs 01, 12, 23 and loads, have been found and laid off in position as shown. From some convenient point O, draw a line parallel to $m\ n_3$, the direction of the thrust S at the crown acting at m,

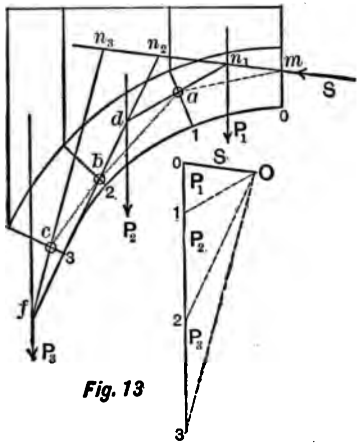


Fig. 13

and to the scale of force make the length Oo equal to S.

From the point *o* thus found, draw a vertical line and on it lay off successively $01=P_1$, $12=P_2$ and $23=P_3$; also connect the points 1, 2 and 3 with O by straight

lines. These lines are called *rays*, O is the *pole* and the figure is known as the *force diagram*. The rays O1, O2, O3, represent the resultants on the joints 1, 2, 3, respectively in magnitude and direction. To find their position on the arch, produce the thrust S at *m* to intersection n_1 with P_1; from this point, draw a line parallel to ray O1 of the force diagram, intersecting joint 1 at *a*; produce this line on to intersection *d* with P_2, at which point draw a parallel to ray O2, intersecting joint 2 at *b*; again produce the last line on to intersection *f* with P_2, at which point, draw a parallel to ray O3 to intersection *c* with joint 3. The points *a*, *b*, *c*, are the centres of pressure on joints 1, 2, 3 and a broken line connecting *m* with *a*, *a* with *b* and *b* with *c* (dotted line) is the line of resistance. This method of combining each resultant thrust on a joint, with the weight of next voussoir and load to find the resultant on the next lower joint, is open to the objection that any error made is carried on, whereas, by previous methods, any error made in construction is confined to

the joint in question. The polygon $m\ n_1\ d\ f$ is called an *equilibrium polygon*, and we note that it does not coincide with the line of resistance; also that it passes through the centres of resistance a and b, but does not pass through c; hence an equilibrium polygon may pass out of the arch ring at certain points and yet the centres of resistance on the joints be found in the arch ring; so that it cannot be used alone in testing the stability of an arch ring, particularly for semi-cicular arches, though it is generally sufficiently near the truth for flat arcs.

If we produce $f\ d$, which gives the direction of the pressure on joint 2 to intersection n_2 with $m\ n_1$ produced, we have n_2, a point in the vertical through the centre of gravity of the part of the arch from the crown to joint 2. This is true, because we know that we must combine S at the crown, with the weight of arch from O to 2 (P_1+P_2) at such a point on the line of action of S as to give a resultant on joint 2 that will coincide in position and direction with the former

resultant $d\ f$. The only point that satisfies this condition is n_2 which proves the statement. Similarly $f\ c$ produced to intersection with $m\ n_1$ at n_3 gives a point in the vertical passing through the centre of gravity of first three voussoirs and load.

Conversely, if by previous methods we compute the horizontal distances of n_1, n_2, n_3, from the crown and thus fix the points n_1, n_2, n_3 in position, and then find the resultants on joints 1, 2 and 3 by drawing lines from n_1, n_2 and n_3 parallel to rays O1, O2, O3, to intersections a, b and c with joints 1, 2 and 3, these resultants produced to intersection will form sides of the equilibrium polygon $m\ n_1\ d\ f$.

This principle will be utilized in applying the theory of the solid arch further on.

(13*a*.) Sometimes the position of the thrust at the crown is not given, but the thrust at the lower joint 3 (ray O3) is given, passing through c. In this case, P_1, P_2 and P_3 having been laid off in position and the force diagram constructed, draw from c a line parallel to ray O3, to intersection f with P_3; then a line parallel to ray O2,

from f to intersection d with P_2; next a line from d parallel to ray O1, to intersection n_1 with P_1; lastly, draw from n_1 a line parallel to ray $Oo = S$, to intersection m with crown joint; giving thus the same equilibrium polygon $m\ n_1\ d f$ as before, and the same line of resistance $m\ a\ b\ c$ determined as above.

14. Special Properties of the Equilibrium Polygon.

Let Fig. 14 represent an arch, or a portion of an arch, of any kind, loaded in any manner, the joints through A and B being any two joints whatsoever, and the joint through I being *any* intermediate joint.

Call W the weight of arch and load included between joints A and B, W′ and W″ the weights of arch and load included between joints A and I and joints I and B respectively, $l' =$ horizontal distance from the vertical through the centre of gravity of W′ to point I. Assume the thrust upon the joint at I as having the direction Z I H and at the point H, where it intersects the vertical through the centre of gravity

of W′, draw H C for an assumed direction of the thrust upon the joint at A, the point

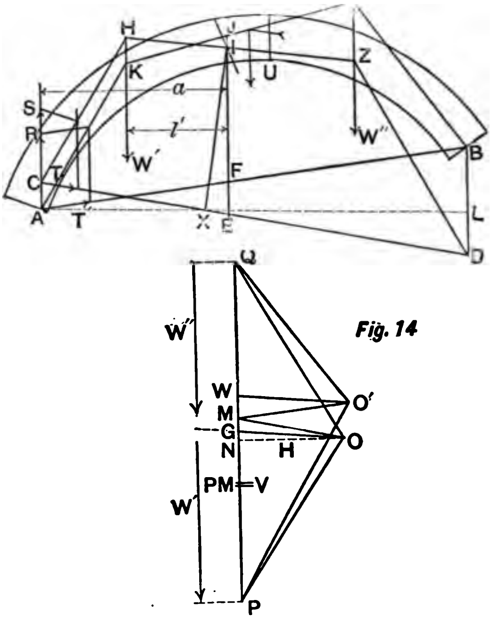


Fig. 14

C lying upon the vertical through A and either above or below it, or in fact coinciding with it as a special case.

In the force diagram below let the vertical Q P, to any scale, represent W, Q G = W″, and G P = W′; then on drawing a line through P, parallel to H C, and a line through G parallel to H I, their intersection gives the pole O; so that drawing through Z a parallel to Q O to intersection with the vertical through B, we establish the point D. Connect C and D by a straight line, called the *closing line*, the vertical through I meeting it at E and the perpendicular let fall from I upon it meeting it at X. Next, through the pole O, draw a line O M parallel to C D to intersection M with Q P and call the length O M = T and M P = V. These lengths represent the two components of the resultant P O which acts along the line C H in true position. The two resultants, whose intensities are P O and O Q, acting up along the lines C H and D Z respectively, support the weight of the arch, or with the components of the weight of the arch, form a system in equilibrium.

At the point C decompose the left reaction into two components, one vertical

acting from C towards S at a horizontal distance a from the vertical through I, and the other acting from C towards D.

The intensities of these two components are given by the lines PM=V and OM=T in the force diagram, though they are represented in the figure above to a diminished scale.

Now if we suppose the part of the arch to the right of the joint through I to be removed and its action on the left part to be replaced by the resultant pressure it exerts at I (=O G of force diagram) acting to the left, the left part of the arch, extending from joint I to A, is evidently in equilibrium under the action of this force, the weight W′ of this part AI with load and the two components V and T at C.

Hence taking moments about I of these forces in equilibrium, we have

$$Va-W'l'=T.\ \overline{IX}.$$

In the force diagram, the line ON=H, the pole distance, is the horizontal distance from O to PQ.

Now the moment T. $\overline{IX}$ is equal is H.

$\overline{IE}$; since, if at the point E we decompose T (acting along CD) into vertical and horizontal components, the moment of the former about I is zero and the moment of the latter$=H.\overline{IE}$, which is thus equal to the moment of T about $I=T.\overline{IX}$, whence,

$$Va-W'l'=H.\overline{IE}.$$

Similarly if we conceive drawn another equilibrium polygon A K J Y B, with pole O′, passing through the points A, J and B in the verticals through C, I and D respectively, we should have

$$V'a-W'l'=H'\,\overline{JF},$$

where V′ is the component AR directed vertically, T′ the one along AB, of the left reaction acting from A towards K and H^1 is the new pole distance.

The resultant whose line of action is KJY, is the resultant on the joint passing through I or the previous joint considered, J being simply a point on that resultant in the vertical through I, but not on the *joint*, unless the latter is vertical. It follows that W′ is unaltered and it remains to be proved that $V'=V$.

We have, from the first equilibrium polygon the system of forces V and T acting at C, the weight of arch and load W acting l to left of vertical B D, and lastly the resultant reaction at D, all together constituting a system of forces in equilibrium. Hence taking moments about D, we have ($\overline{AL}$ being horizontal)

$$V.\ \overline{AL}=Wl.$$

Similarly, from the second equilibrium polygon, we see that the forces V′, T′, W and the reaction at B, constitute a system of forces in equilibrium; hence taking moments about B, we have

$$V\ .\overline{AL}=Wl$$

since the right members are equal in the last two equations, it follows that $V=V'$; whence the left members of the two equations preceding the last two are equal and we have

$$H.\ IE=H'.JF\ \therefore\ H'=H\frac{IE}{JF}$$

or the pole distances vary inversely as the ordinates IE and JF from I and J to the closing lines CD and AB.

The above conclusions equally hold if the weights W', W'' are replaced by their components, representing the weights of successive voussoirs and loads, and the equilibrium polygon is drawn for the entire system, since the pressures on joints A, I and B are not altered by the decomposition; whence the previous formulas all hold and the principles in question are established as before.

The reader who is acquainted with the principles of the equilibrium polygon will recognize that C H Z D is the equilibrium polygon for the *vertical* forces V (=PM at C, W' at H, W'' at Z and a vertical force=MQ in intensity, at B.

Also the formula above,

$$Va-W'l'=H.\ IE,$$

proves that the moment of all the vertical forces to the left of a point I is exactly equal to the pole distance H measured to the scale of force, multiplied by the vertical distance from I to the line CD (known as the closing line) measured to the scale of distance.

Similarly since $V' = V$, the same moment is equal to $H'.JF$ as shown above or the principle is generally true for any equilibrium polygon drawn as above.

As in drawing a line of resistance it is generally most convenient to know the thrust on the vertical joint U at the crown in position, direction and magnitude, the method of finding it for the polygon whose pole is O′, will now be indicated. At the intersection of JY with the vertical through the centre of gravity of the portion of the arch included between joints U and I with its load, combine the thrust at J ($= G O'$ in force diagram), acting from left to right, with the weight of the portion considered $= \overline{WG}$ on force diagram, giving the thrust at the crown $= \overline{O'W}$ in magnitude and direction. We find it in position by drawing through the above intersection a line parallel to $\overline{O'W}$ as as shown by the small arrow. From this thrust at the crown we draw the line of resistance right and left of the crown in the usual manner.

If the joint through I is the vertical

crown joint, then KJY is at once the line of action of the thrust there, its magnitude being equal to O′G in the force diagram.

Where the joint through I is near A, it may happen, particularly when the tangent to the centre line of the arch ring near A is nearly vertical, that the vertical through the centre of gravity of W′ or the weight from joint I to joint A with load, lies to the left of A.

On constructing the figure, however, it will be found that all of the previous equations hold, so that the above conclusions are generally true.

The principle proved above, enables us to draw an equilibrium polygon through any three points, as A, J and B of an arch.

To do this take I (Fig. 14) on a vertical through J, draw the joint through I and find the values and positions of W′ and W″.

Then, as detailed above, draw the trial equilibrium polygon C H I Z D corresponding to pole O, the points C and D being in the verticles through A and B.

On drawing from O, OM ∥ DC to inter-

section M with QP and $MO' \parallel AB$, a distance to the right whose horizontal projection is

$$H' = H \frac{IE}{JF},$$

the new pole O′ is established.

The equilibrium polygon A K J Y B can now be directly drawn, beginning at A, J or B at pleasure, as it must pass through these points. Thus beginning at J, we draw $KY \parallel O'G$ to intersections K and Y with W′ and W″; then lines parallel to PO′ and O Q through K and Y respectively should pass through A and B.

15. *Example.*

In Fig. 15 is snown an arch of 75 feet span, 15 feet rise and 7.5 feet depth of keystone, the top of the backing rising to 2 feet above the top of arch ring. The specific gravity of this backing is supposed to be 0.8 of the masonry of the arch ring, and the heights above the arch ring are reduced to eight-tenths of the original on both sides of the arch, though it is

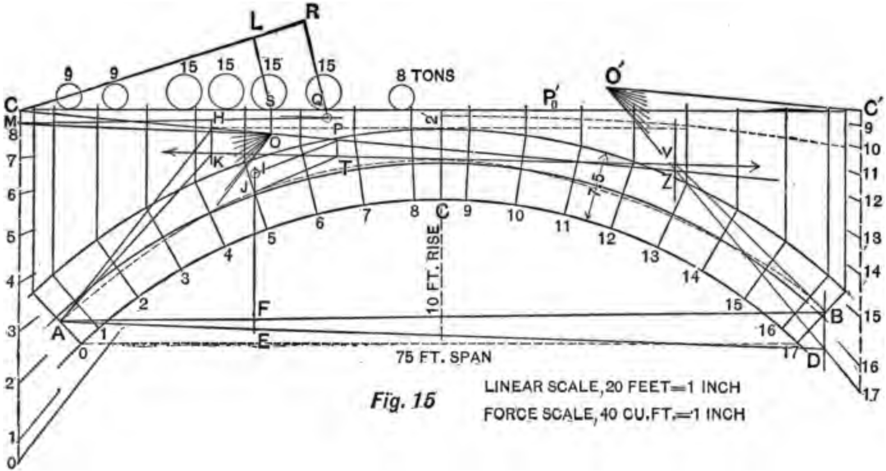


Fig. 15

LINEAR SCALE, 20 FEET=1 INCH

FORCE SCALE, 40 CU.FT.=1 INCH

only shown on the right half to avoid confusion of lines. Cooper's class "extra heavy A" locomotive is placed over the left half in the position shown. The weight on each pair of wheels will be supposed to bear on the length and width of a cross tie, and to be transmitted vertically downwards. If we regard the cross ties as 8 feet in length, the weight per foot of length for drivers, is 15 ÷ 8 short tons which is equivalent in weight to 26.8 cubic feet of stone, weighing .07 ton or 140 lbs. per cubic foot. This supposes the ring stones to be made of stone of this specific gravity which corresponds to good sandstone masonry.

The equivalent for the pilot and tender wheels are 14.3 and 16.1 cubic feet respectively. These weights, resting on the successsive voussoirs through the spandrels, are given in column *s*, Table II (upper numbers), their distances from the crown are given in column *c* and their moments about the crown in column *m*. The lower numbers pertaining to any joint, in columns *s* and *m* refer to the un-

loaded half and are found from Table I by summing up the corresponding quantities for any joint which are made out as explained in Art. 11.

It will be observed that the division of the arch ring is different from that hitherto used. As we shall use this division in a subsequent article, from considerations pertaining to the theory of the solid arch, we shall state that the centre line of the arch ring is divided into 32 equal parts (2.77 ft. long each), and then the voussoir joints are taken in succession two divisions apart, except for the two voussoirs next the springing and the two next the crown where only one division is taken. This division of the arch ring is easily made with dividers. The volume of each small voussoir is $2.77 \times 7.5 = 20.8$ and the large ones have a volume $= 2 \times 20.8 = 41.6$ cubic feet. The horizontal distances from the centre of each voussoir (taken on centre line) to the crown are the lowest numbers of column c, Table I. The columns S, M and C in either table are made out as usual (see Art. 11).

If it is desired to pass a line of resistance through the points A and B at the springing joints and through J on the 5th joint, we first draw the load line C 876 ... 0 for the left half of arch and lay off on it from Table II, column S, C 8 = 25, C 7 = 92, etc.; also from column C lay off the successive distances, on a horizontal (dotted) line through the crown, to the verticals through the centres of gravity of the loads from the crown to any joint. Similarly C′ 9, . . . , 17 is the load line for the right half and Y Z is a vertical through its centre of gravity.

We now assume the thrust at the crown to act along same line H Z and draw rays CO and C′O′ parallel to it; at the point H where this line meets the vertical through the centre of gravity of the left half of the arch and load, draw H A. Then draw a ray from *o* in left load line, parallel to H A to intersection with ray C O at O, thus fixing the pole O. Make C′ O′ = C O to fix the right pole, since C O gives the magnitude of the thrust at the crown.

The thrust at crown meets Y Z at Z,

from which point draw a parallel to ray O′–17 to intersection D with the vertical through B. Connect A, D and A, B and mark the points E and F where they are intersected by a vertical through J.

At the point where the trial crown thrust meets the vertical through the centre of gravity of arch and load from crown to joint 5, 10.82 ft. to left of crown (Table II) draw a parallel to ray O5 to intersection I with the vertical through J.

Through the trial pole O (as in Art. 14) draw $OM \parallel AD$ to intersection M with load line; then draw $MP \parallel AB$ a distance to the right whose horizontal projection is, $H' = H\frac{IE}{JF}$. As the distances I E and J F are rather short, double them and lay of along same line C R through C, $CL = 2 \times JF$, $CR = 2 \times IE$. Then if S is the intersection of a horizontal line through C and a vertical through O, connect L and S and draw R Q parallel to L S to intersection Q with C S produced, which construction gives $H' = CQ$. A vertical through Q to intersect M P at P gives the new pole

P. At the right, draw ray C′ P′ parallel and equal to a ray from P to C (not drawn) and we have the new pole P′ at the right.

We have only now to draw through J a line parallel to ray P5 to intersection T with the line of action of the weight from the crown to joint 5, to fix a point T in the line of action of the new thrust at the crown.

Through T draw a line parallel to PC, and from the intersections K and Y of this line with the verticals through H and Z draw lines parallel to rays Oo and O′–17 respectively, which lines should pass through A and B if the work has been done correctly.

If this does not obtain the error is perhaps largely due to not taking off the lengths IE and JF (with dividers, never with scale) with sufficient accuracy. At any rate, the construction must be repeated if necessary until the line of resistance will pass through the three points A, B and J.

The centre of resistance on *any* joint is found in a similar manner to that already given for the springing joint B. The

broken line connecting the centres of resistance on all the joints is shown by the dotted line to pass near the centre line throughout, which is due to the great depth of arch ring chosen in this case for the sake of a clear figure.

When the point J is on the crown joint, take I to coincide with it and draw the line of action HZ of the trial thrust at the crown through I=J. The construction then proceeds as before, only the final thrust at the crown, K Y, is now drawn through I=J parallel to ray PC.

The construction is thereby simplified for this case.

The graphical methods given above of determining completely the thrust at the crown for a line of pressures passing through any three points in the arch ring will probably be preferred to the analytical methods of Art. 10.

The preceding article indicates the entire construction for the case represented by Fig. 15. Let the reader repeat this construction on a scale of 3 or 4 feet to the inch and compare with the numerical values of the components of the thrust at the crown found by the formula method.

As another example, in the arch shown by Fig. 9 (Art. 10) use the quantities in the Tables of Arts. 8 and 9 and

pass a line of pressures through the lower middle third limit at the left springing and through the upper middle third limits at joint 2 on the left and at the springing joint on the right. With a scale of 30 feet to the inch it is found that the thrust at the crown acts 0.76 foot below the centre of the joint, its horizontal component being 294 cu. ft., vertical component 18 cu. ft.

At joints 1 and 2 on the right the centres of resistance fell 0.14 and 0.12 respectively below the middle third limit.

CHAPTER III.

CURVES OF RESISTANCE CORRESPONDING TO THE MAXIMUM AND MINIMUM HORIZONTAL THRUST. UNIT STRESSES.

16. PROPERTIES OF CURVES OF RESISTANCE.

In Fig. 16, *e c a* represents an unsymmetrical arch, or an arch acted on by forces not symmetrical, vertical or inclined.

Let P=resultant of the external forces

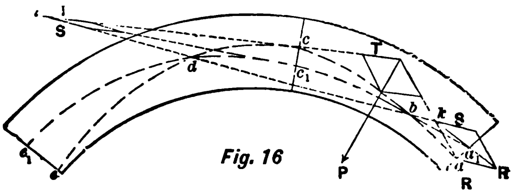


Fig. 16

acting on the arch between *a* and *c*, not including the reaction R at *a*. Then on combining $R=\overline{ak}$ with P, we get the centre of pressure *c* on the joint cc_1. Similarly we could proceed for other points, *b*, *d*, *e*, of the curve of resistance, corresponding to the resultant $R=\overline{ak}$, acting through *a* in the direction $\overline{ak}$.

Let a_1 b c_1 d e_1 be a second curve, corresponding to the reaction R′ at a_1. Now *if* S *is such a force, acting towards the left, that when combined with* R, *it gives* R′ *as a resultant*, we can find a point c_1, on joint $c\,c_1$, of the new curve of resistance, either by combining R′ with P as before, or by combining its components with P: thus call the resultant of R and P, T; this combined at l with S, gives a resultant which cuts joint $\overline{c\,c_1}$, at c_1, *a point lying between* $\overline{kl}$ and c, $\overline{kl}$ being in the direction of s produced.

By this construction it is seen that the new curve of resistance, corresponding to the reaction R′ at a', *passes through b and d*, the points where $k\,l$ intersects the first curve of resistance; for other joints, as $e\,e_1$, the new curve lies nearer $\overline{kl}$ than the first curve; since when S acts to the left, the combination of T, for any joint, with S, gives a resultant acting *between* T and S, which therefore cuts the joint nearer $\overline{kl}$ than the first center of pressure.

The above supposes that neither R nor S are vertical, but that both act to the

left, whence the horizontal component of R′ exceeds that of R. The joints are, moreover, not supposed inclined more than 90° from the vertical, counting from the top.

17. *Prop.* *If two curves of resistance cut each other, the curve which lies nearest the straight line, which joins their common points, corresponds to the greatest horizontal thrust.*

We have seen in the preceding article that the two curves *can only* intersect on the straight line $k\,l$ (Fig. 16) as implied in the proposition.

Now if, at any joint $c\,c_1$, the centre of pressure c_1, corresponding to the curve $a_1\,b\,c_1\,d\,e_1$, lies nearer $\overline{kl}$, the straight line joining b and d, than the curve $a\,b\,c\,d\,e$, then we may suppose a force S, acting in the direction $\overline{kl}$, to be combined with T at l, to effect it. The force S, thus found, must therefore, when combined with R at a give R′; since R and S produce the same effect as R′; so that all points of the first curve can be found by combining R with the resultant of the force P, up to

the joint, and afterwards combining their resultant with S.

The force S, acting to the left, increases the horizontal component of the resultants on each joint; hence the curve $a_1\, b\, c_1\, d\, e_1$ corresponds to a greater horizontal thrust than the curve $a\, b\, c\, d\, e$, as stated in the proposition.

If the arch is symmetrical, the curves of pressure are symmetrical with respect to the crown, whence $k\, l$ must be horizontal.

18. (1). *If a curve of resistance has two points common to the intrados and an intermediate point common to the extrados, it corresponds to the minimum horizontal thrust.*

For, suppose the curve $a\, b\, c\, d\, e$, Fig. 16, touches the extrados near c, the intrados on both sides nearer the abutments.

Then any other curve of resistance, $a_1\, b\, c_1\, d\, e_1$ that remains in the arch ring, must cut the first, only in points on the straight line $\overline{kl}$, joining any two points of intersection.

Now the new curve, near the points of contact of the first curve with the con-

tour curves of the arch ring, must, if it remains in the arch ring, pass nearer $\overline{kl}$ than the first curve, whence, by *Prop.* Art. 17, the first curve corresponds to a less horizontal thrust. Q.E.D.

(2). *If a curve of resistance, a b c d e, Fig. 17, has two points of contact, b and d, with the extrados, and an intermediate point of contact c with the intrados, it corresponds to a minimum horizontal thrust, if b c d, in the vicinity of c, lies*

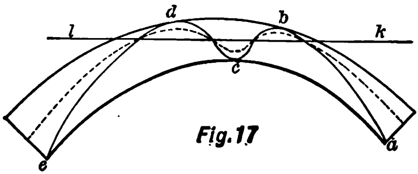


Fig. 17

between the intrados and the straight line $\overline{bd}$.

For any other curve, lying in the arch ring, as the dotted curve, *must* lie nearer the straight line *kl*, joining their points of intersection, than the first, in the vicinity of *b c* and *d*, and thus corresponds to a greater horizontal thrust. This case of

the minimum is rarely or never found in practice.

(3). *If, however, the intrados, in the vicinity of c lies between the curve b c d, Fig.* 18, *and the straight line* $\overline{bd}$, *the curve corresponds to a maximum horizontal thrust;* since this curve lies nearer $\overline{kl}$ than any other, as the dotted curve of resistance.

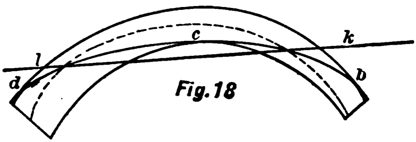


Fig. 18

It is seen that $\overline{kl}$ in Fig. 18 lies above $\overline{bd}$, whereas the reverse occurs in Fig. 17.

When a curve of resistance possesses both the properties of the maximum and minimum of the thrust, the arch is at the limit of stability; as see all the figures relating to the experiments in the Appendix. The above principles were first stated by Dr. Herman Scheffler.

19. When the arch is symmetrical (and symmetrically loaded), the curves of resist-

ance are symmetrical with respect to the vertical through the crown; hence for a half arch and load we can state the following propositions:

(1). *When the point of contact with the extrados is higher than the point of contact with the intrados, the curve of resistance corresponds to the minimum horizontal thrust.*

(2). *When the point of contact with the extrados is lower than the point of contact with the intrados, the curve of resistance corresponds to the maximum horizontal thrust.* It generally touches the intrados at the crown joint.

The curve corresponding to the minimum thrust generally touches the extrados at the crown for usual loads and depths of arch ring; but for thin arch rings with little or no surcharge above the crown, especially with gothic arches, the curve of resistance passes below the crown and touches the extrados at some lower point. This likewise may happen, even for segmental arches of usual depth, when a heavy load is placed over the middle of either

haunch, as shown in Art. 9 for the curves drawn corresponding to the minimum horizontal thrust within the middle third limits.

For segmental arches, the lower point of contact for either maximum or minimum thrust is generally at the springing joint.

20. The conclusions above hold equally when we wish to find the maximum or

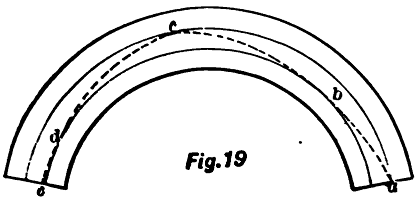


Fig. 19

minimum thrust for curves contained within the middle third or any other limits, only we substitute the upper and lower limiting curves for extrados and intrados in the enunciations.

In Fig. 19 the dotted line represents a curve of resistance corresponding to the maximum and minimum of the thrust at the same time, within the limits shown.

The part *a b c* corresponds to the max. and

b c d to the min. of the thrust within those limits; for *b* lies above a straight line drawn from *a* to *c*, Art. 18 (3), and *c* lies between *b* and *d*, Art. 18 (1).

In an arch by itself, if but one curve of resistance can be drawn within its contour curves, thus corresponding at once to the maximum and minimum thrust, the arch is at the limit of stability.

Unit Stresses at any Point of a Joint.

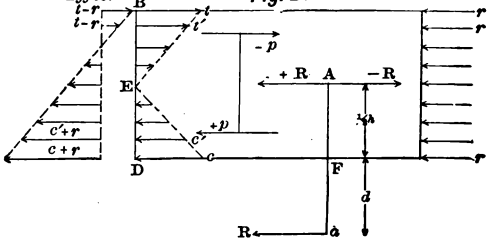
21. In Figure 3 of Article 5, the resultants of the molecular stresses on the joints ab, a_1 b_1, a_2 b_2, are Q, R_1, R_2, respectively. To find these stresses at any point of a joint as a_2 b_2. The resultant R_2 on the joint a_2 b_2 meets it at A_2, whose distance from the nearest edge (a_2 A_2 in this case) we shall call *d*. The joint is rectangular in shape; its width perpendicular to the plane of the paper being unity, and its radial length a_2 b_2 we shall call *h*.

As R_2 is generally inclined to the normal to the joint, resolve it at A_2 into two components, the first, which call R, being normal to the joint and the other acting

parallel with the joint. The last component is a shearing force and undoubtedly lessens the resistance of the joint, but because its influence is difficult to estimate it is generally neglected.

It will first be assumed that the mortar in the joint can withstand both tension and compression, in which case R_2 can fall outside the joint a_2 b_2 (Fig. 3), a distance *d*

Fig. 20



measured along the line of the joint produced, and still stability be assured if the mortar possesses sufficient strength. Similarly for concrete arches. In Fig. 20, let B D represent the joint a_2 b_2 of Fig. 3 and at the point where the resultant on this joint cuts the joint or the joint produced, resolve it into shearing and normal compo-

nents; neglecting the former, we have the normal component R, acting through *a*, a distance $d + \frac{1}{2} h$ from the centre E of the joint (Fig. 20).

At the point E conceive two opposed forces $+R, -R$, equal and parallel to R to act. This does not destroy equilibrium. To avoid confusion $+R$ and $-R$ are drawn through A, a point in the normal to BD at E, but it is understood that A is supposed to coincide with E. The force R with the force $-R$ forms a right-handed couple RR, that can be replaced by the equal couple $\overline{pp}$ or the forces, $t, t' \ldots . c', c$, equal and opposed to the uniformly increasing tensible *resistances* from E to B and the compressive *resistances* from E to D, E lying in the centre of gravity of the cross-section.

From the theory of flexure, calling $t = c$ the stress per square unit at the extreme edge B or D, we have the moment of the couple $RR = M = R\,(d + \frac{1}{2} h) = \frac{1}{6}\, ch^2$; whence,

$$t = c = \frac{6\,R\,(d + \frac{1}{2} h)}{h^2} \quad \ldots\ldots (1).$$

The remaining force + R at E (A), acting at the centre of gravity of the joint corresponds to a uniform compression,

$$r = \frac{R}{h} \quad . \quad . \quad . \quad . \qquad (2)$$

over the whole joint (shown by the little arrows to the right, though really acting along B D).

As the forces given by (1) and (2) act at the same time, their algebraic sum or "effect" (see figure) gives the actual stresses along joint B D, which are thus seen to be uniformly increasing from a neutral axis, not passing through E. The tension at B $= t - r$ and compression at D $= c - r$.

As in voussoir arches the resultant rarely or never passes outside the arch ring, *let us suppose hereafter* that it cuts the joint to which it refers, in its interior a distance *d* from the nearest edge; then we have,

$$t = c = \frac{6 R (\frac{1}{2} h - d)}{h^2} \quad ;$$

$$r = \frac{R}{h} \quad ;$$

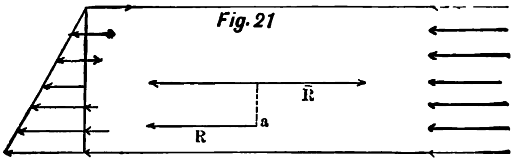
whence,

$$t - r = 2\left(1 - \frac{3d}{h}\right)\frac{R}{h} \quad . \; . \; . \; . \quad (3).$$

$$c + r = 2\left(2 - \frac{3d}{h}\right)\frac{R}{h} \quad . \; . \; . \; . \quad (4).$$

As this theory supposes that the limit of elasticity has been nowhere exceeded, the stretch or shortening of the "fibres" is proportional to the stress and therefore to the distance from the neutral axis; hence

Fig. 21



a plane section before strain remains a plane section after strain as in the ordinary theory of beams.

It is seen from (3) that $t - r$ is positive, or tension is exerted at B, for $d < \frac{1}{3}h$; for $d = \frac{1}{3}h$, $t - r = o$ or there is no stress at B and the stress at $D = c + r = 2\frac{R}{h}$ or double the average (see Fig. 21);

lastly, for d > ⅓ h, the stress on the joint is compressive throughout (Fig. 22).

Hence when the resultant cuts the joint within its middle third there are only compressive forces exerted on the joint; when it passes outside the middle third, tensile

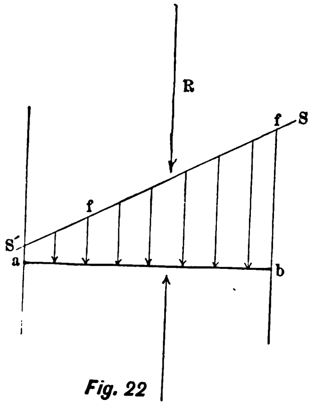


Fig. 22

forces are brought into play if the mortar can supply them.

In the last case, if the mortar cannot resist tensile forces, the normal component of the resultant will be decomposed into stresses, over a length of joint 3d, pro-

portional to the ordinates of a triangle and the part of the joint beyond this length (3d) will open.

This is plain, because we have just found that for $d = \frac{1}{3} h$, that $3d = h$, was under compression, the stress at the farthest edge from the resultant being nothing.

The stress at the most compressed edge is likewise double the average on the part of the joint under compression $\therefore$ it is

$$2 \frac{R}{3\,d}$$ for all cases where the resultant lies

in an outer third of the arch ring and the mortar cannot resist tension.

The last formula and formula (4) suffice to give the maximum compression per square unit for the cases to which they refer, where the voussoirs are each in one block. In case the arch ring is made of several rolls as in brickwork or when we meet with joints transverse to the radial joints as we proceed along the latter, the above theory is only approximately true.

Example 1. In an arch of 5 feet radial length of joint at the springing, the resultant has a normal component of 73.36 tons and it acts 1 ft. below the centre of the joint.

(1). What is the stress at the intrados if the mortar cannot resist tension?

Answer, $\frac{2\times73.36}{3\times1.5}=32.6$ tons pr. sq. ft.

(2). What is the stress at the intrados and extrados if the mortar can resist tension?

Here $R=73.36$, $h=5$ ft., $d=1.5$ ft.; hence by eqs. (3) and (4),

$$\text{Stress at extrados}=2\left(1-\frac{3\times1.5}{5}\right)\frac{73.36}{5}=2.9,$$

$$\text{Stress at intrados}=2\left(2-\frac{3\times1.5}{5}\right)\frac{73.36}{5}=32.2$$

tons per square foot.

Example 2. . In the example above, if the resultant acts 0.5 ft. above the centre of the joint, find the unit stresses at the intrados and extrados.

22. In the theory of the solid arch, which will be presently referred to, it is necessary to know the moment M of the resultant on any joint with respect to the centre of that joint. This may be expressed in three different ways: first, by the product of the resultant by the perpendicular from the centre of the joint upon it; second, by $R(\frac{1}{2}h-d)$ as above, since the shearing component has no moment, and lastly by multiplying the horizontal component of the resultant on any joint by the vertical distance from the centre of the joint to where a vertical through this

centre meets the resultant. This is plain, since at the intersection of the vertical with the resultant decompose the latter into vertical and horizontal components. The moment of the former about the centre of the joint is zero; hence the moment of the latter (the constant horizontal thrust of the arch) is equal to that of the resultant itself.

23. Method of Failure of Arches.

In the appendix will be found an account of a number of experiments on small wooden arches at the limit of stability with their corresponding resistance lines, which, of course, correspond to the maximum and minimum thrust at the same time, within certain limits. (See art. 20.)

In the fourth experiment, *with a yielding pier*, the top of the pier and the haunches of the arch moved outwards and the crown descended. In this case the limiting line of resistance (for the slightly deformed arch) touches the extrados at the crown, the intrados at the haunches and

could be inscribed within the middle third drawn to the lower joints of rupture. The lower joints of rupture made angles with the vertical of 67° in the first case and about 62° in the second.

If we call $s =$ span, $h =$ rise, and $k =$ depth key, the following table gives Woodbury's limits of k in terms of the span for *segmental arches* of various ratios of h to s; the first set of values $k \div s$ giving the ratios of depth of key to span which permits only one line of resistance to be drawn within the limits of the arch ring; the second set of values, the

$\frac{h}{s}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{10}$
$\frac{k}{s}$	$\frac{1}{47}$	$\frac{1}{50}$	$\frac{1}{54}$	$\frac{1}{60}$	$\frac{1}{65}$	$\frac{1}{78}$
$\frac{k}{s}$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{27}$	$\frac{1}{30}$	$\frac{1}{33}$	$\frac{1}{39}$

ratios of k to s in order that only one line of resistance can be inscribed within the middle third of the arch ring. If the spandrel is carried above the crown, these

ratios will become less; but, if after the centres are struck, the spandrels are brought to a level with the top of key-stone, the last ratios should certainly never become less or joints will open. In fact, if these values are attained the construction for the solid arch will give a line of resistance passing slightly outside of the middle third and thus bringing tensile stresses on fresh mortar at some of the joints. Properly, the spandrels should be built up progressively from key to abutment, so that the height at the key is attained before that at the abutment. As it will be well for the reader to test some of these values, it may be mentioned that for the first set of values above, the line of resistance touched the intrados at the crown, the extrados at the haunches, and the intrados at the springing. The curves limited to the middle third touched the lower limit at the crown and springing and the upper limit about the haunches.

Exercise.—When the span is 100 feet, rise 20 feet, and depth of key 4 feet ($^1/_{25}$ space), and the spandrel rises to a

level from the top of key, construct the single line of resistance.

The method of failure of segmental arches with *rigid abutments and an eccentric load over the haunch of the left half* may be illustrated by a reference to fig. 19. Conceive the arch ring to diminish in depth so that finally but one curve of resistance can be drawn therein. It will be found to touch the *extrados* to the left of the crown and at the right springing joint; it will touch the *intrados* at the left springing joint and a little to the right of the crown.

In a large arch, crushing at the edges is experienced before this minimum depth of arch ring is attained. In any case the arch will sink at the joints under the load, the joints at the intrados opening, whereas the arch ring rises a little to the right of the crown, since the pressure there is nearly all concentrated at the lower edge. At the left springing, the lower part of the arch rotates downwards about the lower edge and at the right springing about the upper edge of the joint. As a consequence,

the arch divides into three parts; the left part falling inwards, the middle portion rising at the right but falling at the left end and the right segment rotating upwards about the upper edge of the right springing joint.

In any kind of an arch, loaded in any manner, the method of failure is easily arrived at by simply studying the line of resistance pertaining to the case.

CHAPTER IV.

Line of Resistance Determined as in a Solid Arch. Method of Isolated Loads for Segmental Arches. Computation of Depth of Keystone.

24. A great many approximate solutions have been proposed for the voussoir arch, but none satisfactory. The true line of resistance in an arch depends primarily upon its elasticity, and likewise upon the care with which the stones are cut and fitted, the thickness and yielding of the mortar joints, the settlement and time of striking of the centres if the mortar joints have not hardened, and finally the yielding of the piers or abutments. So many of these influences cannot be exactly estimated that the author has hesitated about applying the theory of the solid arch "fixed at the ends" to the voussoir arch, particularly on account of its complexity, though in Van Nostrand's Magazine for January and November, 1879, he claimed that the theory

of the solid arch was the most exact solution for the cases assumed, and a graphical treatment was given in the last named article, to which reference will be made further on.

From Prof. Swain's article in Van Nostrand's Magazine for October, 1880, it is to be inferred that the application of the theory of elasticity to the stone arch had already been considered by a few authors mentioned. In 1879 Winkler published his notable theorem (given in the article last mentioned); also Castigliano applied the theory of the solid arch, after the method of "least work," to stone arches. In the same year Prof. Greene, of the University of Michigan, published a more practical treatment, founded on the analytical theory of the circular arch, using the method of isolated loads.

In this chapter a method similar to Prof. Greene's is used, the tables, however, being obtained by aid of Winkler's tables, for the solid arch "fixed at the ends." The *span* of the centre line of the arch ring was divided into equal parts, and the quan-

tities in the tables found for isolated loads at the points of division by interpolation (by aid of diagrams to a large scale), from Winkler's constants, which refer to equal angular divisions, though a few direct computations were made for loads very near the abutment. Winkler's theory and tables for the solid arch are given in Du Bois's "Graphical Statics," and the preliminary computations of c_1 and c_2 were made out as explained in "Theory of Solid and Braced Arches" by the writer, p. 90, the deformation of the arch ring, due to bending moments, being alone considered. The general table given in this chapter refers only to arches whose rise is one fifth of the span.

The method of *equal horizontal divisions* adopted here offers great practical advantages, and enables one with small labor, comparatively, to investigate the strength and stability of a given stone arch. *The position of the live load* causing maximum departures of the centres of pressure from the centres of the joints, is more accurately ascertained than hitherto, and some unexpected conclusions were established.

25. The theory of the solid arch "fixed at the ends," is strictly applicable to a solid arch of stone, iron or other material perfectly fitted, when not under stress, to rigid abutments, the theory requiring three conditions to be satisfied, viz.: (1) the tangents to the centre line at the springs are fixed in position, (2) span invariable, and (3) the vertical displacement of one springing above the other equals zero; all having reference to the deformation of the arch ring due to the stresses caused by its own weight and any load that may be applied.

The theory is evidently rigidly applicable to a voussoir arch, *with no mortar in the joints*, provided the voussoirs are cut so perfectly that the arch fits accurately between the rigid abutments when not under stress. When thin cement mortar is used in the joints and allowed to harden before the centres are struck, the conditions are but little altered; but for bad fitting stones and thick mortar joints, not sufficiently hardened, neither this theory nor any other

is exactly applicable, though it may be regarded as some guide in the relative dimensioning of arches of various spans.

It must be carefully observed, too, that, if in a voussoir arch, without mortar, fitting perfectly when not under stress between the skewbacks, the centre of pressure on any joint, as determined by the theory of the solid arch, falls without the middle third, the joint will only bear on a length equal to three times the distance from the centre of pressure to the nearest edge of the joint (Art. 21). In this case only the bearing surface of the joint must be included in the formulas for fixing the resistance line. Hence a new determination has to be made on this basis and so on, until the assumed and computed bearing joints agree. This method, although possible, is very tedious, so that the theory is only a practicable one when the arch ring is such that the true line of the centres of pressure lies everywhere within the middle third.

Only such arches are examined in this work, besides space does not permit the

consideration of any arch rings except those of uniform cross-section.

26. In Fig. 23, the circular curve, of rise equal to one-fifth of its chord or span, represents the centre line of an arch ring of constant section. The span is 5 units in length, the rise one unit. Each *half span* is divided into 10 equal parts and vertical lines drawn through the points of division. Where the successive lines cut the curve will be designated as points 0, 1, 2, 3, . . . , 10 and 1′, 2′, 3′, . . . 10′ for the left and right halves respectively, the left springing point being called 10, the right springing 10′, the crown 0 and the numbers increasing regularly along the arc from the crown to the two spring-joints respectively.

If a single load W is placed on the arch, supposed to be without weight, its equilibrium polygon will consist of two straight lines. When W is directly over a point of the arch previously fixed, as 4, the table gives at once c_1, y and c_2 in terms of h = rise of centre line of arch considered, where c_1 = vertical distance from point 10

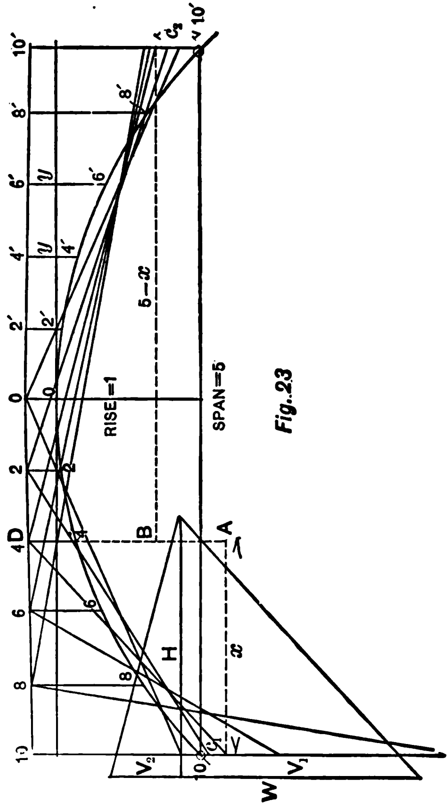


Fig. 23

on arc to side of equilibrium polygon, c_2 the same for point 10′ on arc and y = vertical distance above the crown to apex D of equilibrium polygon.

Thus if W acts at 4 on arch, the rise of whose centre line (above the centre of the springing joints) is 10, the span 50, we have from table, that the resultant at left abutment acts $-.16 \times 10$ or 1.6 below centre of joint, the resultant at right springing, acts $+.296 \times 10$ or 2.96 above centre of joint, and the apex D lies $.216 \times 10 = 2.16$ above the highest point of centre line of arch ring, plus coefficients corresponding to points above the arc and minus below, as just indicated. On laying off W on a vertical line just to left of arch as shown, and drawing lines from the extremities parallel to the sides of the equilibrium polygon passing through D, the intersection gives the pole of the force diagram. The length of the horizontal line from the pole to the line representing W, gives H = horizontal thrust due to W alone, and the line divides W into the two vertical components of the reactions V_1

GENERAL TABLE.—Rise $=\frac{1}{5}$ Span.

Point	C_1	Y	C_2	H	V_1	V_2	$M_1=Hc_1$	$M_2=Hc_2$	M_1+M_2
0	+.152	.224	+.152	1.1639	.5000	.5000	+.1769	+.1769	+.1769
1	+.106	.222	+.197	1.1512	.5710	.4290	+.1221	+.2268	+.3489
2	+.035	.220	+.236	1.0861	.6437	.3563	+.0380	+.2564	+.2944
3	−.055	.218	+.268	.9807	.7134	.2867	−.0539	+.2628	+.2089
4	−.160	.216	+.296	.8473	.7772	.2228	−.1356	+.2508	+.1152
5	−.310	.213	+.321	.6867	.8366	.1633	−.2129	+.2204	+.0075
6	−.535	.210	+.344	.5098	.8896	.1104	−.2727	+.1754	−.0973
7	−.900	.206	+.365	.3327	.9342	.0658	−.2994	+.1214	−.1780
8	−1.640	.200	+.384	.1706	.9691	.0309	−.2798	+.0655	−.2143
9	−3.800	.191	+.402	.0497	.9918	.0082	−.1888	+.0200	−.1688
	h	h	h	W	W	W	hW	hW	hW

and V_2 at left and right springings respectively. For $W = 1$ (to a large scale) we can thus find graphically the coefficients in columns H, V_1 and V_2, but it is better to compute them from easily derived formulas:

$$V_1 = \frac{A\,D}{x} H \quad ; \; V_2 = \frac{B\,D}{5-x} H;$$

$$H = W \div \left(\frac{A\,D}{x} + \frac{B\,D}{5-x}\right)$$

We have here, $A\,D = 1 + y - (c_1)$,
$B\,D = 1 + y - (c_2)$,

paying attention to the sign of c_1. Since x is known, H can be at once computed and afterwards V_1 and V_2. The coefficients thus found are to be multiplied by W in any application, as indicated in the table.

The moment about any point of the centre line of the arch ring, for any weight W is equal to the H corresponding, multiplied by the vertical distance from the point to the equilibrium polygon corresponding to W (see Art. 22). The algebraic sum of such moments due to any number of weights, gives the total moment

at the point and the sum of the H's gives the total horizontal thrust due to the weights. If the moment at the point, due to the weight of the arch is found and added to the preceding moment and its H likewise added to the previous sum of the H's, then the quotient of the total moment divided by the total H, gives the vertical distance from the point on the centre line of the arch ring to the equilibrium polygon corresponding to the weight of arch and loading considered. This is the method to be used in fixing any point of the equilibrium polygon after the theory of the solid arch "fixed at the ends". When the moment is plus, the equilibrium polygon is above the centre line of the arch ring at the point; when minus, below.

For convenience, the values of $M_1 = Hc_1$ and $M_2 = Hc_2$ (the moments at the left and right springing joints) as well as $M_1 + M_2$ are given in the general table. The coefficient for a weight at the crown (point 0 of arc) is only half of $M_1 + M_2$ for reasons that will appear directly.

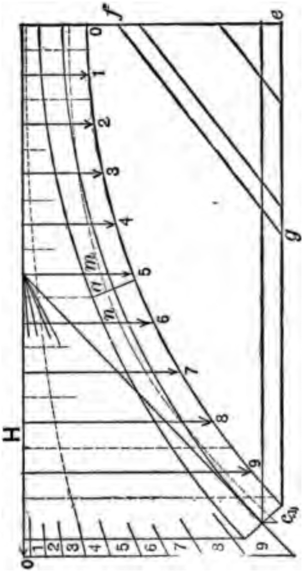


Fig. 24

Let fig. 24 represent any arch whose rise is one-fifth the span; the rise of its centre line is also one-fifth its chord from centre to centre of springing joints. Divide the half chord of the centre line into ten equal parts and erect verticals at the points of division. Reduce the length of

the part of these verticals comprised between the extrados of the arch and the horizontal roadway line to eight tenths of each. This is best done graphically. Thus if $e\,g$ is laid off equal to 10 and $e\,f$ equal to 8 to any scale, and we lay off any length from e along $e\,g$ and from its extremity draw a parallel to $g\,f$ to intersection with $e\,f$, the distance from this intersection to e gives the reduced length. The material above the arch ring to the reduced contour can then be regarded as weighing the same per cubic foot (.07 ton) as the arch ring. On drawing dotted verticles half way between the first verticals, the area of the trapezoids comprised between any two successive dotted verticals will be equal to the width multiplied by the length of the full vertical between them, and it will be regarded as a force acting along the medial (or full) vertical. In fact, for a length of arch equal to unity, this area is also the volume of a prism having for a base this area, and by multiplying by .07 it can be reduced to tons.

In all the computations below the por-

tion of the arch from either springing joint to the first dotted vertical will be neglected, as its influence is very small in fixing the true equilibrium curve. The weight at the crown is that due to the portion comprised between the adjacent dotted verticals on either side. This division of the arch is different from that hitherto used.

27. Let us proceed now to the consideration of an arch of 100 ft. span. The rise is 20 ft. and the depth of the key 5 ft., the horizontal roadway rising 2 ft. over the crown. The first column in the table below gives the joint of the arch at which the weight is concentrated. The "depth" of a "trapezoid" (column 2) multiplied by the constant "width" 5.2 (column 3) gives the area = volume = W, expressed in cubic feet. We have only to multiply these values, as well as those given in columns H and M, by .07 to reduce to tons when desired.

The coefficients of columns H and $M_1 + M_2$ are taken from the general table above. On multiplying these by the successive values of W we get the horizontal

thrusts and moments given in columns H and M_1 of the table. Any thrust in column H is that due to the load at the point corresponding, hence the sum gives the total thrust due to loads 0, 1, 2, - - - , 9. On adding to this the same sum, less that due to the load at the crown, we get the thrust due to the weight of the entire arch $= 594.9$ cubic feet, since the loads at equal distances from the crown are the same. Similarly M_1 for load at $6 + M_1$ for load at 6^1, is the same as M_1 for load at $6 + M_2$ for load at $6 = (M_1 + M_2)$

Table for 100 ft. span, 20 ft. rise, 5 ft. depth of Key.

Point	Depth	Width	W	H Coeff	H	M_1+M_2 Coeffs.	M_1
			cu. ft.		cu. ft.		
0	6.6	5.2	34.3	1.1639	39.92	+.1769	+ 6.07
1	6.8	"	35.4	1.1512	40.75	+.3489	+12.35
2	7.2	"	37.4	1.0864	40.63	+.2944	+11 01
3	8.	"	41.6	.9807	40.80	+.2089	+ 8.69
4	9.1	"	47.3	.8473	40.08	+.1152	+ 5.45
5	10.5	"	54.6	.6867	37.49	+.0075	+ 0.41
6	12.2	"	63.4	.5098	32.32	—.0973	— 6.17
7	14.4	"	74.9	.3327	24.92	—.1780	—13.33
8	17.2	"	89.4	.1706	15.25	—.2143	—19.16
9	20.3	"	105.6	.0497	5.25	—.1688	—17.83
			583.9		317.41		— 12.51
			549.6		277.49		20
Weight of arch			= 1133.5		H = 594.9		M_1 = —250.2

for load at 6. The same principle holds for any two loads at equal distances from the crown. The coefficient for the load at the crown was not doubled, as there is no other load corresponding to it.

On adding up the figures in the last column and multiplying by $h = 20$, we find the total moment $= -250.2$, and on dividing this by the total $H = 594.9$ we find that the resultant at the left springing passes 0.42 foot below the centre of the joint. The same holds at the right springing on account of symmetry. The equilibrium polygon can now be drawn, as the vertical component $= \frac{1}{2}$ weight of arch $= 566.75$ and $H = 594.9$ are given as well as $c_1 = -.42$. Construct the force diagram by drawing H and from its left extremity; lay off successively on a vertical downwards half the load at crown and the loads at 1, 2, - - - , 9 (see fig. 24). We draw the equilibrium polygon from a point .42 below the centre of the left springing joint, as explained in Art. (13a). It is very near the centre line and is shown approximately in fig. 24. Its vertical distance from points

6, 5, 0 and 3′ on the arc are respectively +.27, +.2,—.25 and — 1; so that multiplying by H = 595, the moment at these points are + 161, + 120, — 149 and — 60 respectively. We have previously seen (Art. 13) that the resultant along *m n*, say of fig. 24, is strictly that pertaining to the joint between *m* and *n* at *a*, the true resistance *curve* passing slightly below *m*; still for *purposes of comparison* below it is near enough to consider the line *m n* to represent the line of action of the resultant acting on the joint passing through the point where the vertical through *m* cuts the centre line, particularly as we shall find that the maximum departure of the line of resistance from the centre of the joints, when the live load is considered, is nearly always at a springing joint where no error is made. Further, as the resultants on the upper joints are nearly perpendicular to them for usual loads the intersection of a perpendicular from *m* on the joint corresponding can be regarded as the centre of pressure, for purposes of comparison below.

We are now prepared to consider the additional influence of the live load, which in all the subsequent examples in this chapter will be taken as a locomotive load of 6000 pounds per foot of track, 20 feet in length or slightly greater or less, corresponding to the horizontal divisions of the arch, followed by a tender load of 2400 lbs. per ft. 30 ft. long, about, and this followed by another locomotive load as before. This about corresponds to Cooper's class extra heavy A, without the pilot wheel. For cross ties 8 ft. long, these loads for a slice of the arch 1 foot thick, are 750 and 300 lbs. per ft. respectively. As the horizontal divisions of the arch are 5.2 ft. each, the locomotive load on each division $= 750 \times 5.2$ lbs., or the weight of 27.8 cubic ft. of stone weighing 140 lbs. per cubic ft.; the tender load on each division is 11.1 cubic ft. stone. The first locomotive will be assumed to cover 4 divisions of 5.2 ft. each or 20.8 ft.; the tender 6 divisions or 31.2 ft. *As M and H both vary, for any point* as we shift the live load, it is only *by trial*

that the maximum value of $M \div H = c$, can be found for the point considered.

Thus consider *point* 5 of centre line of arch ring where, for dead load only we have found $M = +120$ and $H = 594.9$. On a large scale drawing similar to fig. 23 (except that loads at all points 1, 2, 3 - - - are considered), on measuring the vertical ordinates from point 5 of arc to the equilibrium polygons corresponding to weights $W = 27.8$ at 4, 5, 6, 7, and adding the results $(= .778)$ we have the total moment due to locomotive load at 4, 5, 6, 7, $= .778 \times h$ $W = .778 \times 20 \times 27.8 = +432.6$. That due to tender load at 8, 9, is similarly $= .061 \times 20 \times 11.1 = +13.5$; adding moment $= +120$ due to dead load, the total moment at point $5 = +566.1$. The horizontal thrust due to loads 27.8 each at 4, 5, 6, 7, is found by adding the coefficients in column H of general table for points 4, 5, 6, 7, and multiplying by 27.8 $\therefore 2.3765 \times 27.8 = 66.1$. Similarly for tender load at points 8, 9, the thrust is $.2203 \times 11.1 = 2.4$. Adding the thrust due to weight of arch, 594.9 to the sum of these two and we have

the total horizontal thrust = 663.4. On dividing + 566.1 by this, we find that the equilibrium polygon due to dead load, locomotive load at 4, 5, 6, 7, and tender load at 8, 9, passes 0.89 foot above the centre of the joint through point 5 on arc, whence on drawing a normal to the joint through the point found, the resistant is found to pass 0.8 above centre measured along the joint. The live load may now be moved to right or left one or more divisions, but for no other position is the resultant on joint 5 so far from the centre as is found by a computation similar to the above. A similar investigation for *point* 6 with locomotive loads at 4, 5, 6, 7 and tender loads at 8, 9, gives M = +520, H = 664, so that c = + .78 or less than for point 5. From fig. 23 it is evident that for points 5 or 6 the live load should not extend as far as point 2 from the left, as the moment due to a load at 2 or to the right is negative. Trial shows that for a maximum *c* at 6 the load should not extend to 3, but from 10 to 4 inclusive as just given.

At the *crown* the max. departure is caused by loads, say from 4 to 10 and 4^1 to 10^1; but it is not so great as elsewhere, and as the loading is unusual it will not be further considered.

At point 3^1 of arc, for loc. loads at 0, 1, 2, 3, tender at 4 to 9 inclusive, $c = -370 \div 746 = -.5$ or less than for any other point hitherto examined. This value will doubtless be increased slightly by moving live load one or two divisions to the left.

Consider next the *right springing joint.* M_2 and consequently c_2, for dead load is minus. This obtains for spans of about 35 ft. and upwards, for rise $= 1/5$ span, so that the live load to left of crown giving a positive moment at 10^1 acts against the dead load; hence we should not expect to find c_2 for such spans as great as c_1 at the left springing, where the moments due to both live and dead loads have the same (minus) sign, but only trial can determine. For spans less than about 35 ft., max. c_2, may be greater than max. c_1, as in fact was found to be the case for an arch of 25 ft. span.

For this 100 feet span, $M_2 \div H$ was computed for the front of the loc. load at 2′, 1′, 0 and 1 successively, and found to be greatest when the loc. loads were at 0, 1, 2, 3 and tender loads at 4, 5, 6, 7, 8, 9. The computation proceeds as before, the sum of the coefficients in columns M_2 and H of the general table, for the points above being multiplied by hW and W respectively, thus:

$$
\begin{array}{lr}
+ .923 \times 20 \times 27.8 & = + 513.2 \\
+ .853 \times 20 \times 11.1 & = + 189.4 \\
\text{Dead Load M't} & = - 250.2 \\
\hline
\text{Total,} \qquad M_2 & = + 452.4
\end{array}
$$

$$
\begin{array}{lr}
4.382 \times 27.8 & = 121.8 \\
2.597 \times 11.1 & = 28.8 \\
\text{Dead load thrust} & = 594.9 \\
\hline
\text{Total,} \qquad H & = 745.5
\end{array}
$$

$$c_2 = + 452.4 \div 745.5 = + .61$$

The computation for *max.* c_1 proceeds after the same principle. The maximum c_1 corresponds to loc. loads at 5, 6, 7, 8 and tender load at 9. The moment coefficients are taken from column M_1 in general table.

$$
\begin{array}{lr}
-1.0648 \times 20 \times 27.8 = & -592.0 \\
-\ \ .1888 \times 20 \times 11.1 = & -\ \ 41.9 \\
 & -250.2 \\
\hline
\text{Total,} \qquad M_1 = & -884.1
\end{array}
$$

$$
\begin{array}{lr}
1.6998 \times 27.8 = & 47.25 \\
.0497 \times 11.1 = & .55 \\
 & 594.90 \\
\hline
\text{Total,} \qquad H = & 642.70
\end{array}
$$

$c_1 = -884.1 \div 642.7 = -1.376$, or the resultant passes 1.38 ft. below the centre of the left springing joint for this position of the live load.

To draw the left resultant in position we next compute the vertical component V_1. Add up the coefficients in column V_1, of general table for points 5, 6, 7, 8 and multiply by 27.8; also multiply (for tender load at point 9) .9918 by 11.1; the sum added to the half weight of arch gives the total V_1

$$
\begin{array}{lr}
3.6295 \times 27.8 = & 100.8 \\
.9918 \times 11.1 = & 11.0 \\
\tfrac{1}{2} \text{ weight arch} = & 566.7 \\
\hline
\text{Total,} \qquad V_1 = & 678.5
\end{array}
$$

From the point, 1.38 ft. below the centre of the left springing joint, lay off vertically upwards $V_1 = 678.5$, to any scale, and from the upper extremity of this line draw a horizontal equal in length (to the scale of V_1) to total $H = 642.7$ to fix the pole of the force diagram. A line from the pole to the lower extremity of V_1 gives the magnitude and direction of the resultant on the left springing joint. It cuts this joint .95 ft. below its centre. For accuracy the left springing joint should be drawn to a large scale (anywhere along the radius), and this construction made to the large scale as this is found to be the joint where the centre of pressure is farthest from the centre.

As the resultant passes 0.13 ft. outside of the middle third the depth of key must be increased. From this and some other examples it was found that if the depth of key was increased by 3 to 4 times the departure, the line of resistance for the new arch would lie inside the middle third limit, just touching it at the critical joint, the left springing in this instance.

In this case the depth of key was increased by $4 \times 0.13 = 0.52$ ft., or say 0.5 ft., so that the new key was 5.5 feet. A new construction and computation for this arch showed that the resultant on the left springing joint passed .02 ft. inside the lower middle third limit or practically touched it.

If desired, the equilibrium polygon for the entire arch can now be drawn for the last loading considered. As before we compute V_1, H, c_1, and it is best to compute c_2 as a check on the construction which can be made as explained in Art. 13a. It is much better, however, to find by computation the position of the centre of gravity of the left half of arch and load and determine by construction at once the centre of pressure at the crown joint, from which points the polygon can be drawn more accurately towards either abutment.

28. On investigating arches of various spans in the manner indicated above, it was found that the springing joints were the only ones necessary to examine, except in the case of a 12.5 ft. span, 2.5 ft. rise

and 2.2 ft. depth of key, where a single concentrated load of 40,000 pounds over point 6 was found to cause the resultant on the joint through 6 to reach the upper middle third limit. The load in no other position gave as great a departure on any other joint. In this case the values of c, V_1 and H were computed, and from the force diagram resulting the true direction of the resultant on joint 6 in position and magnitude was obtained.

In any arch, the resultant on the critical joint having been found in position and magnitude, the normal component can be scaled off and the maximum intensity of stress at the most compressed edge found as in Art. 21.

The following table gives the final results of a series of constructions to determine the depth of key for stone arches of rise = one-fifth the span, so that the line of resistance should everywhere be contained within the middle third of the arch ring of uniform cross-section and just touch it at the critical joints.

The arch ring was supposed to weigh

140 lbs. per cubic foot, the material above it 112 lbs. per cu. ft., and the loading as given above, viz.: 6,000 lbs. per foot of track locomotive load for about 20 feet (depending on the length of the horizontal divisions of the arch), followed by a tender load of 2,400 lbs. per foot on about 30 feet, and this followed by a second locomotive and tender of the same weights, with the one exception of the 12.5 ft. span, where a load of 40,000 lbs. on two drivers was alone assumed. These loads were supposed to bear equally on 8 feet cross ties and to be transmitted vertically to the arch.

The actual lengths of locomotive and tender loads measured along the rails for the different spans was as follows:

Span.	Length loc. load.	Do. tender load.
25,	12.00,	none,
50,	15.60,	none,
75,	19.50,	none,
100,	20.80,	5.2,
125,	19.38,	12.92,
150,	23.25,	15.50.

It is only in the case of the 150 ft. span

that the locomotive length appreciably exceeded 20 ft., but the weight of arch was so great, compared with that of the load, that the error in finding the depth of key was small.

TABLE GIVING THEORETICAL DEPTH OF KEY.

Span	Rise	Key	Loc 1 Load at	Tender at	Maximum Intensity of Stress
Feet	Feet	Feet			Tons pr. sq. ft.
12.5	2.5	2.2	6	none	
25.	5.0	2.6	0.1....9	"	9.
50.	10.0	3.5	4....9	"	14.
75.	15.0	4.5	5....9	"	22.
100.	20.0	5.5	5....8	9	25.
125.	25.0	6.25	5. 6. 7	8, 9	30.
150.	30.0	7.	5, 6, 7	8, 9	36.

As mentioned above, the joint of rupture where the departure of the resistance line from the centre of joint was greatest, was found to be the left springing, except for the 25 ft. span, where it was the right springing and the 12.5 ft. span, where 6 was the critical joint.

The above values are plotted to scale ' Fig. 25, being shown by the small circles. A line through these circles is nearly straight, but not sufficiently so for accuracy. The following table gives these

and interpolated values for every 5 feet for use of constructors:

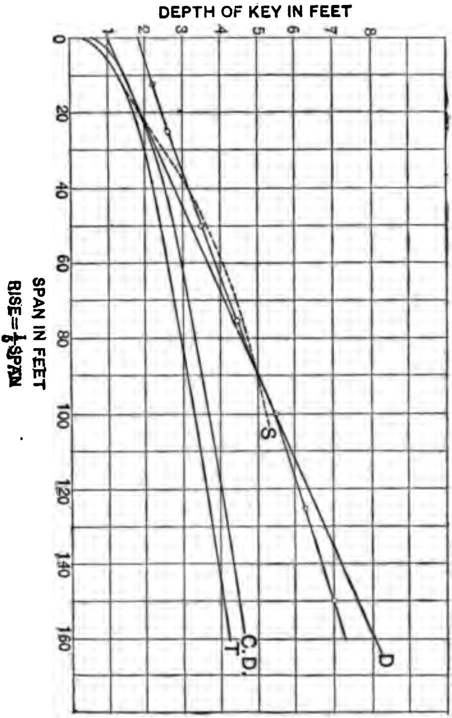


Fig. 25

DEPTH OF KEY FOR ARCH OF UNIFORM SECTION AND RISE $= \frac{1}{5}$ SPAN, ALL DIMENSIONS BEING IN FEET.

Span	Key	Span	Key	Span	Key
5	1.96	55	3.7	110	5.80
10	2.12	60	3.9	115	5.95
12.5	2.20	65	4.1	120	6.10
15	2.27	70	4.3	125	6.25
20	2.43	75	4.5	130	6.40
25	2.60	80	4.7	135	6.55
30	2.77	85	4.9	140	6.70
35	2.95	90	5.1	145	6.85
40	3.13	95	5.3	150	7.00
45	3.30	100	5.5	155	7.15
50	3.50	105	5.65	160	7.30

On fig. 25 are plotted for comparison the depths of key purposed by Dejardin (line D), Scheffler (line S, by interpolation from his tables for rise $= \frac{1}{4}$ and $\frac{1}{6}$ span), Croizette-Demoyers (line C D) and Trautwine (line T) (see Art. 31).

The depths of key, as computed, are in excess of most of the values given, all of which refer to materials of only average strength (second class masonry for the Trautwine line). This excess was to be expected, for most of the old formulas are founded on the successful practice of the past, and cannot therefore be expected to give results corresponding to the very much heavier locomotive loads of to-day, though

some of them may be a rude sort of a guide in the design of common road bridges.

The French authors quoted, after finding the depth of key (as plotted above), then increase the radial length of joint towards the abutment, making it vary as the secant of the inclination to the vertical.

The above table is for an arch of uniform section. The same material can be better disposed, perhaps, by making the depth of key less and increasing the length of joint as we approach the abutment; but the theoretical treatment of this case falls under that of the arch of variable cross-section, and it has to be omitted for want of space.

The table gives the depth of key for immovable piers or abutments and arch stones that *fit perfectly* between the skew-backs, when laid on the centres (without mortar) and supposed not under stress. Where the *abutments or piers are yielding*, either from not having a rock foundation or from too small a width (particularly in a series of arches), or where *mortar* in the joints is used which is not hard

when the centres are struck, or in any case when the mortar joints are not thin and hard, so that the arch cannot be regarded as practically homogeneous throughout, then an increase should be given to the depths above, according to the best judgment of the engineer.

The effect of temperature on the arch has been omitted though its effects are large (see Science Series, No. 48). The dynamic effect of the load is supposed to be taken up by the spandrels.

The arch ring has been supposed above to be of sandstone and weigh 140 lbs. per cubic foot. If it weighed 160 lbs. the same depth of key would correspond to live loads $^8/_7$ of those assumed, or a less depth of key would suffice for same loads.

29. Maximum intensity of stress allowable in stone and brick arches.

The average pressure on a joint is equal to the normal thrust divided by the area of the joint, and this reaches a maximum in existing bridges (according to Scheffler,

for masonry weighing 150 lbs. per cu. ft.) from 17 tons per square ft. at the crown to 62 tons per square ft. at the springing. In such arches the resultants on the joints act outside the middle third; but even if they acted *at* the middle third limits, the intensity of pressure at the most compressed edge would be double the above or .34 and 124 tons per square ft. at crown and springing respectively. Scheffler recommends *not exceeding* average pressures at the crown and springing, corresponding to columns of the same material as the voussoirs 207 and 308 ft. high, or say 15.5 to 23 tons per square ft. for stone weighing 150 lbs. per cubic ft.

This rule seems safe (for cement joints, not common mortar) and corresponds for stone weighing 150 lbs. per cu. ft. or fair limestone, to intensities of pressure at the most compressed edges of 31 tons per sq. ft. at the crown and 46 tons at the springing.

30. Existing structures that have done good service, as well as arches which have failed, afford the data from which the many empirical formulas for depth of keystone have been derived. These formulas are not based on theory but on successful practice and are valuable in their way,

but very unsatisfactory in some respects. Thus they differ very greatly in their results, some giving double the depth of Keystone for certain spans as others, and besides they rarely make a distinction between a common road bridge and one intended for the heaviest modern locomotives to pass over at great speed.

In fact these formulas generally represent the practice of the past, mainly for light road bridges (with a few exceptions) and can serve a useful purpose in the design of such bridges; but most of them are evidently inadequate for the heavy moving loads of to-day on railroads.

As preliminary to writing some of the best known of the formulas, it will be convenient to give simple formulas for expressing the radius in terms of the span, for ease of reduction, as most of the formulas are expressed in terms of the radius.

If we call s = span in feet, h = rise in feet of a circular arch, r being the radius of the intrados, we have the well-known relative

$$\left(\frac{s}{2}\right)^2 = h\,(2\,r - h)$$

$$\therefore r = \frac{h}{2} + \frac{s}{8}\left(\frac{s}{h}\right) \ldots . (1)$$

$$\text{For } \frac{h}{s} = \frac{1}{5},\ r = \frac{29}{40}\,s$$

$$\text{"}\ \frac{1}{6},\ \text{"}\ \frac{5}{6}\,s$$

$$\text{"}\ \frac{1}{8},\ \text{"}\ \frac{17}{16}\,s$$

$$\text{"}\ \frac{1}{10},\ \text{"}\ 1.3\,s$$

$$\text{"}\ \frac{1}{12},\ \text{"}\ \frac{37}{24}\,s$$

31. The following formulas, by French engineers, for k = depth of keystone in feet, are taken from DuBosque ("Ponts et Viaducts en Maconnerie") after reducing to English

equivalents. They are intended to apply to best bricks or to stone not so hard as granite, stone of "medium resistance" as BuBosque styles it.

Perronet's formula is intended for every kind of arch, semi-circular, segmental, elliptical or basket handle, and is as follows:

$$k = 1 + .0347 s \ldots\ldots (2).$$

A more recent author, *Dejardin*, gives the following for circular arches:

$$\left.\begin{array}{l} \frac{h}{s} = \frac{1}{2},\ k = 1 + 0.1\ r = 1 + .05\ s \\ \frac{h}{s} = \frac{1}{6}\ \ k = 1 + .05\ r = 1 + .042\ s \\ \frac{h}{s} = \frac{1}{8},\ k = 1 + .035\ r = 1 + .037\ s \\ \frac{h}{s} = \frac{1}{10},\ k = 1 + .02\ r = 1 + .026\ s \end{array}\right\} \ldots\ldots (3)$$

and for elliptical or basket handled arches

$$\frac{h}{s} = \frac{1}{3},\ k = 1 + .07\ r \ldots\ldots (4)$$

in which *r* equals the radius of curvature at the crown.

Another French authority, *M. Croizette Desnoyers*, gives the following for segmental arches, the first also applying to semi-circular arches.

$$\left.\begin{array}{l} \frac{h}{s} > \frac{1}{6},\ k = 0.5 + .27\sqrt{2\ r}. \\ \frac{h}{s} = \frac{1}{6},\ k = .5 + .253\sqrt{2\ r} = .5 + .327\sqrt{s}, \\ \frac{h}{s} = \frac{1}{8},\ k = .5 + .235\sqrt{2\ r} = .5 + .342\sqrt{s}, \\ \frac{h}{s} = \frac{1}{10},\ k = .5 + .217\sqrt{2\ r} = .5 + .35\sqrt{s}, \\ \frac{h}{s} = \frac{1}{12},\ k = .5 + .2\sqrt{2\ r} = .5 + .351\sqrt{s}, \end{array}\right\} \ldots\ldots (5)$$

He likewise adapts the first formula of (5) to elliptical or false elliptical arches of small rise, by considering r to represent the radius in feet of an arc of a circle of same span and rise.

Dupuit gives a much smaller depth of key by the following formulas:

$$\left.\begin{array}{l} \frac{h}{s}=\frac{1}{2},\ k=0.36\sqrt{s}=\sqrt{.13\,s}, \\ \frac{h}{s}<\frac{1}{4},\ k=0.27\sqrt{s}=\sqrt{.073\,s} \end{array}\right\}\ldots\ldots(6)$$

These call for good granite laid with care.

To all the above formulas must be added .02 *H, where H is the height of the surcharge above the crown, reduced if necessary to the density of earth.*

This is plainly a very rude and inexact way of allowing for an extra surcharge of earth, since a moderate amount must add materially to the *stability*, the load being fixed and symmetrical with respect to the crown and thus giving a line of resistance much nearer the centre line of the arch ring than an eccentric rolling load, and not calling for an extra section on account of the dynamic effect of the live load. Actually a smaller arch ring can be used up to a certain height with the same security, especially considering that the active (or passive) pressure of the earth around the arch almost ensures stability (when crushing is not to be feared) even for thin arch rings.

In any case the arch should be examined, first leaving out the horizontal pressure of the earth, which generally only adds to the stability, and afterwards considering it.

After finding the depth of keystone by preceding formulas, the Europeans generally increase the (radial) depth of arch ring from the crown to the springing. If we call d the angle that any joint makes with the vertical, and l the radial length of any joint *for segmental arches*, the following formula for this length is frequently used, $l = k \sec d$ . . . . (7).

Du Bosque prefers to find the radial length of the joint

at the springing joints and crown by (7) and other formulas above, and draw an arc of circle through the upper ends of those joints to define the extrados.

For semi-circular, elliptical or basket handled arches the rule is to measure up from the springing line, half the rise *and* draw a horizontal to intersection with the intrados; the joints drawn there normal to the intrados called "the joints of rupture," must have a length equal to the depth of keystone multiplied by a coefficient, which is

2	for semi-circles	
1.8	" ellipses, &c., rise	$= \frac{1}{3}$ span.
1.6	" " " "	$= \frac{1}{4}$ "
1.4	" " " "	$= \frac{1}{5}$ "

As before, a circle is drawn through the upper ends of the "joints of rupture," and the crown joint for the extrados down to the joint of rupture and there tangents are drawn to this circle to limit the masonry down to the abutment.*

The above lengths of joints at crown and elsewhere are in excess of average English and American practice, which may be partly due to the poorer qualities of stone found in France. We shall now give some English and American formulas.

Rankine's formulas. Find the longest radius of curvaturer of the arch; then the depth of key for a single arch, including tunnel arches in rock or conglomerate, is

$$k = \sqrt{0.12\,r} \quad . \quad . \quad . \quad . \quad (8).$$

For an arch of a series the coefficient of r under the radical should be 0.17; for a tunnel arch in gravel or firm earth 0.27, and in wet clay or quicksand 0.48. The defect

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* See Van Nostrand's Magazine for December, 1883, for illustrations in article by E. Sherman Gould, C. E.

in this formula is the lack of a constant term to give a proper depth of key for arches of small span. It gives smaller values even than Dupuit's formula for the very best materials.

Trautwine's formula, for first-class masonry,

$$k = 0.2 + \tfrac{1}{4} \sqrt{r + 0.5s} \quad . \quad . \quad . \quad . \quad (9)$$

has the constant term, but is wrong in principle, in that for the same span the depth of key increases with *r* or as the rise *diminishes;* whereas the flatter the arch, for the same span, the less should be the key (where crushing is not in question) since a line of resistance can be more easily inscribed within the same limits in a flat arch than in one of greater rise when the key is the same. This last defect characterizes all the formulas given above, but those of Dejardin and Dupuit.

Trautwine increases the depth given above $\frac{1}{8}$ for second class masonry, and about $\frac{1}{4}$ for brick on fair rubble.

CHAPTER V.

PRINCIPLES AFFECTING SOLID ARCHES FIXED AT THE ENDS. LEMMA.

SECOND GENERAL METHOD OF LOCATING THE TRUE LINE OF RESISTANCE.

32. *Fundamental Equations of Solid Arches "fixed at the ends."*

Space forbids deducing the fundamental equations of solid arches, but the reader is referred to the author's "Theory of Solid and Braced Elastic Arches," pages 21 to 36, for a simple development of the theory. The following are the three equations which must be satisfied in order that a line of resistance may be the true one:

$$\Sigma \frac{M \triangle s}{E I} = 0 \; . \; . \; . \; . \; . \qquad (1),$$

$$\Sigma \frac{Mx \triangle s}{E I} = 0 \; . \; . \; . \; . \qquad (2),$$

$$\Sigma \frac{My \triangle s}{E I} = 0 \; . \; . \; . \; . \qquad (3).$$

The first indicates that the end tangents to the centre line of the arch ring are fixed in direction; the second, that the deflection of one end of the arch below the other is zero; and the third, that the span is invariable.

The centre line of the arch ring is supposed divided into a great number of parts, each equal to $\triangle s$; M represents the moment of the resultant about the centre of the joint traversing the middle of the corresponding $\triangle s$, E is the modulus of elasticity for the corresponding voussoir $\triangle s$ long, and I represents the moment of inertia of a plane joint traversing the centre of $\triangle s$, about a horizontal axis passing through the centre of the joint. The origin of co-ordinates is taken at the centre of the left springing joint; x is the

horizontal and y the vertical distance from this origin to the centre of the corresponding $\triangle s$. The summation extends over the entire arch ring.

If we call H the uniform horizontal thrust of the arch for vertical loading, and v the vertical distance from the centre of any joint traversing the middle of $\triangle s$, to the resultant acting on that joint, we have by Art. 22, $M = H v$.

As by the graphical method it would be impracticable to divide the centre line of the arch ring into a *very great* number of parts, we must content ourselves with dividing it into a certain number of parts of appreciable length and find the M, E, I, x and y for the *middle* of each part, which gives a fairly good average and is sufficiently correct in practice.

If we regard the modulus E as constant throughout the arch ring, it may be dropped from the equations.

Similarly, replacing M by H v, H may be dropped as well as $\triangle s$.

Therefore, *for an arch ring of constant cross-section* where I is constant, the three conditions (1), (2) and (3), reduce very simply to

$$\Sigma\,(v) = 0 \;\ldots\ldots\; (4),$$
$$\Sigma\,(vx) = 0 \;\ldots\ldots\; (5),$$
$$\Sigma\,(vy) = 0 \;\ldots\ldots\; (6).$$

In the case of the voussoir arch, if the curve of the centres of pressure, as determined in position by the above equations in a manner to be shown, lies everywhere in the middle third of the arch ring, there will be no tension exerted on any joint, so that the theory of the solid arch exactly applies when there is no mortar in the joints and the stones are cut to fit perfectly.

If, however, the centre of pressure on any joint without mortar lies outside the middle third, only a *part* of this joint is under compression (Art. 21), so that on substituting the I for that *part* in eqs. (1), (2) and (3) for an arch of variable cross section, a nearer approximation can be made by another trial and so on. Eventually the assumed and computed values of

I will practically agree when the line of resistance can be regarded as fixed. In case there is mortar in the joint that can supply all needed tensile resistance, the line of resistance can pass without the middle third without any change in the equations, as the voussoir arch, save for a different modulus for the thin mortar joints, is subjected to the same deformation as the corresponding solid arch, so that its line of resistance is nearly identical. As in well designed bridges the line of resistance should nowhere pass out of the middle third, for any loading, to avoid the possibility of joints opening with the accompanying infiltration of water, as well as to provide a factor of safety, the tentative method above will rarely be needed, and the line of resistance can be at once found from (4), (5) and (6) for the arch of constant cross section.

33. Lemma.

In the constructions needed for establishing the true line of resistance, according to

the theory of the solid arch, we shall have to solve the following problem:

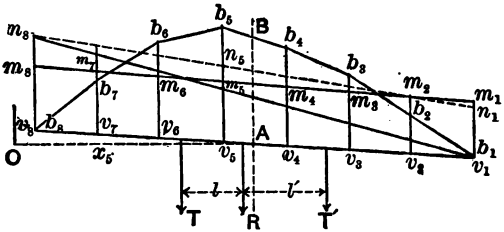

Fig. 26

In fig. 26, having given a series of fixed points b_1, b_2, b_3, ..., b_8, and having drawn parallel ordinates $b_3\ v_3$, $b_4\ v_4$, ..., through them, intersecting a line $m_1\ m_8$ in the points m_1, m_2, ..., it is required to draw $m_1\ m_8$ so as to satisfy the two conditions.

$$\Sigma\ (\overline{m\ b}) = o,\ \Sigma\ (\overline{mb}\ .\ x) = o$$

where $\overline{m\,b}$ represents the ordinate from $m_1\ m_8$ to any one of the points *b*, ordinates above $m_1\ m_8$ being counted positive, those below negative, and *x* represents the distance (abscissa) from an assumed origin O

to the ordinate $\overline{m\,b}$ to which it refers. If we give x the subscript of the ordinate to which it refers, then the above conditions can be written in full, for this figure,

$$m_3\,b_3 + m_4\,b_4 + m_5\,b_5 + m_6\,b_6 - (m_1\,b_1 + m_2\,b_2 + m_7\,b_7 + m_8\,b_8) = o;$$

$$(\overline{m_3\,b_3}\,.\,x_3 + \overline{m_4\,b_4}\,.\,x_4 + \overline{m_5\,b_5}\,.\,x_5 + \overline{m_6\,b_6}\,.\,x_6) - (m_1\,b_1\,.\,x_1 + \overline{m_2\,b_2}\,.\,x_2 + m_7\,b_7\,.\,x_7 + m_8\,b_8\,.\,x_8) = o.$$

Now draw a straight line from b_1 ($= v_1$) to b_8 ($= v_8$) and designate where it intersects the ordinates by v_2, v_3, ...; any ordinates as $m_5\,b_5$ can be written

$$m_5\,b_5 = v_5\,b_5 - v_5\,m_5,$$

or generally,

$$m\,b = v\,b - v\,m\,;$$

which gives m b plus when above $m_1\,m_8$ and minus below, as is imperative. Substituting in the equations above we have

$$\Sigma\,(\overline{vb} - \overline{vm}) = o,\ \Sigma\,(\overline{vb} - \overline{vm})\,x = o.$$

$$\therefore \Sigma\,(\overline{vb}) = \Sigma\,(\overline{vm}),\ \Sigma(\overline{vb}\,.\,x) = \Sigma\,(\overline{vm}\,.\,x).$$

If we call x_0 the abscissa of the resultant of the lines of the type $\overline{vb}$ treated as forces and x'_0 the abscissa of the resultant of the lines $\overline{vm}$ treated as forces, then, since the moment of the resultant is equal to the sum of the moments of the components, the last eq. above can be written,

$$x_0 \Sigma (\overline{vb}) = x'_0 \Sigma (\overline{vm});$$

whence, in view of the relation, $\Sigma (\overline{vb}) = \Sigma (\overline{vm})$, we have $x_0' = x_0$. This establishes the proposition, that when the line $m_1 m_8$ has been determined correctly, the resultant R of the lines $\overline{vb}$, treated as forces, must equal and coincide with the resultant of the lines $\overline{vm}$ treated as forces.

We can quickly, to any convenient scale, find the value of $R = v_2 b_2 + v_3 b_3 + \ldots + v_7 b_7$. Its position can be found by taking moments, most conveniently, about an ordinate AB through the centre of the line $b_1 b_8$.

If the lines $\overline{v\ b}$, by pairs, are equidistant

from AB, as happens in all the applications that follow, call the distances from this (dotted) medial ordinate (AB) to the ordinate through b_1, b_2, b_3 and b_4, z_1, z_2, z_3 and z_4 respectively. Then treating left handed moments as positive, right handed as negative, we have the algebraic sum of the moments about the medial ordinate, equal to $(v_7\ b_7 - v_2\ b_2)\ z_2 + (v_6\ b_6 - v_3\ b_3)\ z_3 + (v_5\ b_5 - v_4\ b_4)\ z_4$; and on dividing this by R (as found above) we have the distance from the medial ordinate to R which can then be laid off in position, as shown in the figure.

The differences as $(v_7\ b_7 - v_2\ b_2)$ can readily be found by taking the distance $v_2\ b_2$ in dividers and laying it off from v_7 along $v_7\ b_7$. The difference between the two lines is to be measured to the same scale as the ordinates $\overline{v\ b}$ in finding the value of R above. (This method is to be generally used in similar cases).

We next draw a trial line $n_1\ n_8$ and divide ordinates as $v_5\ n_5$ (n_5 being the intersection of $n_1\ n_8$ with the ordinate $v_5\ b_5$) in-

to two sets by a line drawn from v_1 ($=b_1$) to n_8.

The resultant T of the sum of the ordinates from the line v_1 v_8 to v_1 n_8 can be found in magnitude, by adding up the ordinates, and in position by taking moments about A as just explained. Lay it off the computed distance to the left of A.

Now the position of T is not changed when v_1 n_8 assumes its true position v_1 m_8 (m_1 m_8 being regarded as the true line to satisfy the original conditions), since all the ordinates in the triangular space v_1 v_8 n_8 are altered in the same ratio. T is thus fixed in position no matter where n_8 may be placed on the line v_8 n_8.

It follows, because of this property and since the ordinates, by pairs, are equidistant from AB, that the resultant T′ of the ordinates intercepted between v_1 n_8 and n_1 n_8 is at the same distance to the right of A that T is to the left. Then if n_1 is afterwards shifted to m_1, T′ is unchanged in position, since all ordinates are altered in the same ratio. Finally, if n_8 is

shifted to m_8, m_1 remaining stationary, the position and value of T′ remain unchanged. Hence lay off T′ in position as far to the right of A as T is to the left, and get its trial value by adding up the ordinates included between v_1 n_8 and n_1 n_8.

From what has been proved above it is plain that if n_1 n_8 has been drawn correctly, the resultant of T and T′ or of the ordinates between v_1 v_8 and n_1 n_8 must coincide with and be equal to R; hence calling l and l^1 the distances from T and T′ respectively to R, we have

$$T = R\frac{l'}{l+l'},\ T' = R\frac{l}{l+l'}$$

If the trial, T representing the sum of the ordinates from v_1 v_8 to v_1 n_8, is not equal to the true value of T just found, reduce the distance v_8 n to v_8 m_8 in the ratio of the true T to the trial T just found.

Change v_1 n_1 to v_1 m_1 in the ratio of the true T′ (given by formula above) to trial T′ = sum of ordinates from v_1 n_5 to n_1 n_8.

The reduction in both cases is best effected graphically.

As the sum of similar ordinates in the triangle $v_1\ n_8\ m_1$ is the same as for the triangle $v_1\ m_8\ m_1$ and their resultant has the same position, it is evident that $m_1\ m_8$ is the true *closing line* (as it is called) to satisfy the conditions.

$$\Sigma\,(\overline{mb}) = o,\ \Sigma\,(\overline{mb}\,.\,x) = o.$$

It is easy to see if the first condition is fulfilled by taking the successive lengths, $m_3\ b_3$, $m_4\ b_4$, $m_5\ b_5$, $m_6\ b_6$ in dividers and adding up along a straight line. Similarly add the lengths, $m_1\ b_1$, $m_2\ b_2$, $m_7\ b_7$, $m_8\ b_8$, along the same straight line. If the two total lengths agree, the condition $\Sigma\,(\overline{mb}) = o$ is satisfied.

When the ordinates $\overline{vb}$ are equal at equal distances from the medial line AB, R must coincide with AB. Now the resultant of $\Sigma\,(\overline{vm})$ cannot pass through the centre unless $m_1\ m_8$ is drawn parallel to $v_1\ v_8$, in which case the lines $\overline{vm}$ will all be of equal length throughout. Their number, in the present instance, is 8, so that by

the first condition $\Sigma\,(\overline{vb}) = \Sigma\,(\overline{vm})$, we have

$$\overline{v_1m_1} = \overline{v_8m_8} = \frac{\Sigma\,(\overline{vb})}{8},$$

which at once determines the line $m_1\,m_8$.

It is equally correct and shorter to take the sum of the ordinates $\overline{vb}$ to one side of AB and divide by 4 when the total number of ordinates is 8.

34. We shall now proceed to design a series of stone (or brick) arch bridges, *whose rise is one-fifth the span*, so that the line of resistance for the position of the rolling load tried shall *just* be contained within the middle third limits of the arch ring, and the intensity of pressure on any edge of a voussoire joint shall not exceed say 30 tons per square foot for the best brick, or 50 tons for good granite or sandstone.

The live load assumed is known in Cooper's Specifications as "*Class extra heavy A*". We give below the distances in feet from the front pilot wheel to each pair of wheels in turn and on the same line, the weight of the pair of wheels in tons of 2000 pounds:

Pair of	Pilot	Wheels	— 0	feet	— 8	tons.
"	Driver	"	— 8.1	"	—15	"
"	"	"	—13.83	"	—15	"
"	"	"	—18.33	"	—15	"
"	"	"	—22.83	"	—15	"
"	Tender	"	—29.92	"	— 9	"
"	"	"	—34.75	"	— 9	"
"	"	"	—40.42	"	— 9	"
"	"	"	—45.25	"	— 9	"
"	Pilot	"	—54.25	"	— 8	"

The position of the pilot wheel of the second locomotive is given last, from which all the wheels of the second locomotive can be located when desired. For short spans the above load may be used or 40 tons equally distributed upon two pairs of drivers, seven feet centre to centre, whichever produces the most hurtful effect.

The arch ring will be supposed of uniform section throughout and composed of brick or sandstone weighing 140 lbs. (.07 tons) per cubic foot. The material above the arch ring up to the level of the roadway will be supposed to have a specific gravity eight-tenths of that of the arch ring.

The load on a pair of wheels is carried through the rails and cross-ties to the bal-

last. If these cross-ties are 8 feet in length we shall suppose the load uniformly distributed along this length, so that for one foot length of cross tie the load will be but one-eighth the load on the whole length of the cross tie. The load for one foot in length of tie is then reduced to cubic feet of masonry by dividing by 0.07.

35. *Example I* (see plate, fig. 27). The figure represents an arch of 12.5 ft. span, 2.5 rise, 2.2 ft. depth of keystone, and radius 9.06 ft., with a surcharge rising 2 ft. above the crown to the level roadway and loaded 3 ft. to the left of the crown with 20 tons on 8 ft. cross ties, equivalent to 35.7 cubic ft. of masonry of the same specific gravity as the arch ring (0.7 ton per cubic foot) on 1 ft. length. The dotted medial lines of the trapezoids are 0.8 of the same lines extended to the roadway, thus giving the "reduced contour" shown. It is evident, for this small span, that the alternative load of 80,000 pounds equally distributed on the two pairs of drivers 7 feet apart, must produce the most hurtful effect. As only one pair of drivers can get

on the half arch they are placed as shown, not quite $^1/_4$ span to left of crown. The centre line of the arch ring is divided into 32 equal parts, the dotted joints are drawn as shown (only a few of these are drawn near the left springing), two divisions apart and radial lines (full lines) are drawn midway between them. The portion of the arch ring between a springing joint and the first dotted joint constitutes one artificial voussoir, and the portion between any two consecutive dotted joints likewise constitutes a voussoir.

As the theory of the solid arch requires that we find the moment for each voussoir about the middle of its centre line, or where the full radial lines cross it, at the points $a_1, a_2, \ldots,$ the construction of Art. 15 applies, or we take the division of the arch included between the *full* radial lines in tabulating quantities, and find the resultants acting on these *full line joints* as in Article 15.

The condensed tables of loads and arms (made out as explained in Art. 15) are as follows:

LEFT SIDE.

Joint	8	7	6	5	4	3	2	1	0
S	1.9	5.7	9.7	49.5	53.9	58.6	63.4	68.5	71.2
C	.24	.76	1.26	2.66	2.77	2.94	3.15	3.42	3.55

RIGHT SIDE.

Joint	9	10	11	12	13	14	15	16	17
S	1.9	5.7	9.7	13.8	18.2	22.9	27.7	32.8	35.5
C	.24	.76	1.26	1.78	2.31	2.82	3.36	3.87	4.13

The loads S are laid off on either side of the arch, the arms either side of the crown, and for an assumed thrust, shown by the upper inclined line through the crown, the various resultants on the joints 1 to 16 are found as in Art. 15. These resultants intersect the verticals through $a_1, a_2, \ldots, a_{16}$ at the points $b_1, b_2, \ldots, b_{16}$.

36. To locate the line k k′, measure the ordinates from a straight line joining a_1 and a_{16}, to the points $a_1, a_2 \ldots, a_8$; add, divide by 8 and lay off the distance vertically from a_1 to k and from a_{16} to k′ and draw the line k k′.

37. Regard $k k^1$ as the trial closing line for points *b;* connect b_1 and b_{16}, also k and

b_{16} by straight lines and find R and the trial T and T′, exactly as explained in Art. 33, in position and magnitude. The positions are given on the figure. Trial T = 23.35, trial T′ = 14.1, R = 38.78;

$$\therefore \text{ true } T = 38.78 \frac{3.07}{5.22} = 22.81;$$

$$\text{true } T' = 38.78 \frac{2.15}{5.22} = 15.97.$$

Hence we must lay off $\overline{b_1 m} = \overline{b_1 k} \dfrac{22.81}{23.35}$

and $\overline{b_{16} m'} = \overline{b_{16} k'} \dfrac{15.97}{14.1}$.

This is best done by the ratio lines as shown, or the distances may be computed and laid off to scale. The line m m′ is thus the true closing line of Art. 33, for points b_1 b_2, . . . b_{16}.

38. The ordinates y_1, y_2, . . ., y_8 (from a_0 a_{17} to a_1, a_2, . . ., a_8) are next scaled off; also the ordinates from k k′ to a_1, a_2, . . ., a_8, called k a_1, k a_2, . . ., k a_8, and we find the value of Σ (k a.y) $= 2\,(k\,a_4 \cdot y_4 + k\,a_5 \cdot y_5 + k\,a_6 \cdot y_6 + k\,a_7 \cdot y_7 + k\,a_8 \cdot y_8 - k\,a_1 \cdot y_1 - k a_2 \cdot y_2 - k\,a_3\,y_3) = 11.50$.

Next, the ordinates from $m\,m'$ to the points b_1, b_1, . . ., b_{16}, called $m\,b_1$, $m\,b_2$, . . ., $m\,b_{16}$ are scaled off and $\Sigma\,(\overline{mb}\,.\,y)$ is found. In the present instance the complete expression for this is $(m\,b_8 + m\,b_9)\,y_8 + (m\,b_7 + m\,b_{10})\,y_7 + (m\,b_6 + m\,b_{11})\,y_6 + (m\,b_5 + m\,b_{12})\,y_5 + (mb_4 + mb_{13})\,y_4 - (mb_3 + mb_{14})\,y_3 - (m\,b_2 + m\,b_{15})\,y_2 - (m\,b_1 + m\,b_{16})\,y_1 = 18.28$, ordinates above $m\,m'$ being treated as positive, those below negative.

We have now only to reduce the ordinates $\overline{mb}$ in the ratio of 11.50 to 18.28 (by the proper ratio lines), and lay off the reduced lengths from the line $k\,k'$ vertically up or down, according to the sign of $m\,b$ to find all the points c_1, c_2, . . ., c_{16} in the true equilibrium polygon for the arch. The ordinate from $m\,m'$ to the point where the trial thrust meets the crown is likewise reduced in the same ratio and laid off from $k\,k'$ to fix the true centre of pressure on the crown joint, .05 ft. below the centre of the joint.

The reader familiar with Prof. H. T. Eddy's "Constructions in Graphical Statics" will recognize that the above pro-

cedure is founded upon his beautiful constructions for the solid arch fixed at the ends.

39. It will now be shown that the points *c*, located as above, are points in the equilibrium polygon, directly over the centre of the artificial voussoirs, which satisfy the three conditions for an arch fixed at the ends (Art. 32)

$$\Sigma\,(ac) = o, \quad \Sigma\,(ac\,.\,x) = o, \quad \Sigma\,(ac\,.\,y) = o.$$

Referring to Art. 36 it is seen that the line k k′ was located in a manner satisfying the conditions,

$$\Sigma\,(k\,a) = o, \Sigma\,(k\,a\,.\,x) = o\;.\;.\;.\;(A),$$

as shown in Art. 33.

Lines of the type *k a* in these formulas refer to vertical ordinates measured from *k k′* to a_1, a_2, ... a_{13}. Similarly *m b* represents a vertical ordinate from line m m′ to b_1 or b_2, etc.; ordinates above *k k′* or $m\,m^1$ being regarded as plus, those below minus.

By the method used in Art. 37 or Art. 33 the line $m\,m^1$ was located to satisfy the conditions,

$$\Sigma\,(mb) = o, \Sigma\,(mb\,.\,x) = o.$$

The ordinates *m b* were now *all* changed in the *same* ratio, which does not affect the position of m m'. The altered ordinates were next laid off from k k', the new value of *m b* being equal to *k c* in the figure.

The conditions just given are thus satisfied by the $\overline{\text{k c}}$' s,

$$\therefore \Sigma\,(\text{k c}) = \text{o}, \quad \Sigma\,(\text{k c . x}) = \text{o} \ldots (\text{B}).$$

Also by the construction of Art. 38, since every *m b* has been altered in the ratio 11.50 to 18.28 to the corresponding *k c*, Σ $(\text{k.c . y}) = 18.28 \times \frac{11.50}{18.28} = 11.50 = \Sigma\,(\text{k a .y})$, as we see by reference to the equations of Art. 38.

If the right member of the last equation is transferred to the left member, since, k c — k a = a c, we have, $\Sigma\,(\text{a c . y}) = \text{o}$. On subtracting eqs. (A) from (B) and writing the equation just found in the group, we have

$$\Sigma\,(\text{a c}) = \text{o}, \quad \Sigma\,(\text{a c . x}) = \text{o}, \quad \Sigma\,(\text{a c . y}),$$

or the points *c* satisfy the conditions for an arch "fixed at the ends."

40. As the closing line m m′ has been shifted to k k′ and the ordinates *m b* altered in the ratio 11.50 to 18.28, by the theory of equilibrium polygons, we draw from the old pole O (on the left) a parallel to m m' (in its first position) to intersection J with the load line, then a horizontal to the right a distance = *old pole distance* $\times \frac{18.28}{11.50}$ to P the position of the true pole for the left force diagram. This is easily effected graphically by laying off J L and J M in the ratio of 11.5 to 18.28 and drawing M P parallel to L I, I being the point where a vertical through O intersects the horizontal through J.

The new pole may likewise be found by the method of Art. 14, by drawing through O a parallel to a line connecting b_1 and b_{16} to intersection with load line, then from this point, a parallel to a line connecting the points c_1 and c_{16}, previously found, a distance to the right whose horizontal projection = old pole distance $\times \frac{18.28}{11.50}$.

To fix the new pole P′ on the right,

draw PC″ equal and parallel to ray P C on left.

The points *c* can be tested by drawing the new resultants on the joints, having given the new poles and the position of the thrust at the crown. These resultants produced to intersection with the respective joints from a_0 to a_{17} give the centres of pressure on the corresponding joints.

The centres of pressure all lie within the middle third of the arch ring, except at joint 6 where the thrust passes exactly $^1/_6$ depth from centre. The intensity at the upper edge of joint 6 is therefore double the mean. The normal component of this thrust is the weight of 63.2 cu. ft. of stone $= 63.2 \times .07 = 4.4$ tons; the mean pressure is thus $4.4 \div$ depth joint 2.2 ft. $= 2$ tons and the intensity at upper edge is therefore 4 tons per square foot.

41. To test the accuracy with which the work has been done, we measure $\overline{ac}$ at the points a_1 to a_{16}, counting distances above *a* plus, below minus; then find the co-ordinates x, y, of each point a_1 to a_{16} regarding a_0 as the origin, x being measured horizon-

tally (along $a_0\ a_{17}$) to the ordinate through any point a and y vertically above $a_0\ a_{17}$ to point a as previously stated, and finally from the products as shown in the table:

Joint	$\overline{ac}$	y	$\overline{ac}.y$	x	$\overline{ac}.x$
4	+ .13	1.89	.246	2.78	.36
5	+ .40	2.26	.904	3.69	1.48
6	+ .38	2.50	.950	4.60	1.75
7	+ .18	2.70	.486	5.57	1.03
8	+ .06	2.80	.168	6.52	.39
15	+ .08	.92	.074	12.90	1.03
16	+ .30	.32	.096	13.66	4.09
	+1.53		+2.924		+10.13
1	— .27	.32	.086	.34	.92
2	— .21	.92	.193	1.12	.23
3	— .07	1.45	.101	1.92	.13
9	— .10	2.80	.280	7.50	.75
10	— .18	2.70	.486	8.48	1.53
11	— .20	2.50	.500	9.41	1.88
12	— .18	2.26	.406	10.34	1.86
13	— .13	1.89	.246	11.23	1.46
14	— .09	1.45	.130	12.10	1.09
	—1.43		—2.428		—9.85

We have here the ratios,

$$\frac{\Sigma(+ac)}{\Sigma(-ac)} = \frac{1.53}{1.43};\quad \frac{\Sigma(+ac\,.\,y)}{\Sigma(-ac\,.\,y)} = \frac{2.924}{2.428};$$

$$\frac{\Sigma(+ac\,.\,x)}{\Sigma(-ac\,.\,x)} = \frac{10.13}{9.85}.$$

These ratios should strictly be unity, so that the conditions of an arch "fixed at the ends,"

Σ (ac) = o, Σ (ac . y) = o, Σ (ac . x) = o, may be satisfied. The results, however, are very close, as will be apparent on supposing the points c all to be lowered one hundredth of a foot only, when Σ (+ a c) will become + 1.47 and Σ (— ac), — 1.52, the minus sums now being the greater. The changes will evidently be equally great in the other sums, so that we have accidentally here determined the centres of resistance within about .01 foot, even on this small scale. We may readily rest content though, with half a tenth of a foot error on each joint owing to unavoidable errors of construction. See the next example where these errors are as pronounced as for any arch examined and yet a shifting of the points c by half a tenth of a foot is about all that is necessary to satisfy the conditions.

We conclude for the arch just examined, for the given position of the live load, that the line of resistence just touches the middle third limit at one point only, and that it possesses the proper margin of safety both as to strength and stability.

42. *Example* II. A segmental stone arch of 25 ft. span, 5 ft. rise, radius 18.12 ft. and uniform depth of arch ring 2.5 ft., was next examined.

The depth of spandrel filling over the crown was 2 ft. and the live load consisted of the last three driving wheels of the locomotive above specified on the left half of the arch, no load on right half.

The loads are given precisely as follows: 15 tons 3.8 ft. from crown, 15 tons 8.3 ft. and 15 tons 12.8 ft. from crown.

The division of arch ring and the construction generally was exactly like that just given for Example I, so that it is not necessary to enter into it. The true thrust at the crown, after the theory of the solid arch, was found to act .06 ft. below the centre of the crown joint and its horizontal component was 120.25 cu. ft. $=8.4175$ tons, the vertical component 10.5 cu. ft. $=0.735$ ton.

If we call for brevity, the distance $\overline{ac}=v$, we find in this instance,

$$\frac{\Sigma\,(+v)}{\Sigma\,(-v)}=\frac{1.90}{1.53};\quad \frac{\Sigma\,(+vx)}{\Sigma\,(-vx)}=\frac{25.64}{18.64};\quad \frac{\Sigma\,(+vy)}{\Sigma\,(-vy)}=\frac{6.32}{5.10}$$

If we conceive the thrust at the crown lowered 0.1 ft. but maintaining its same direction and magnitude, the points *c* will all be lowered 0.1 foot, and the new ratios will be as follows:

$$\frac{\Sigma\,(+v)}{\Sigma\,(-v)}=\frac{1.26}{2.30},\quad \frac{\Sigma\,(+vx)}{\Sigma\,(-vx)}=\frac{16.88}{28.75},\quad \frac{\Sigma\,(+vy)}{\Sigma\,(-vy)}=\frac{2.96}{8.02}$$

Here the minus terms exceed the plus terms so much that it is plain that the thrust has been lowered too much; in fact (neglecting any possible tilting) it is evident that the true position of points *c* lies between the first and last positions and is nearer the former than the latter; so that we can safely say that the first series of points is certainly within 0.05 foot of the correct position.

As the discrepancy above was as great as that found in any case examined, it was thought worth while to show that

the extreme limit of error in any case was negligence and that the construction affords practically exact results.

On constructing the centres of pressure on all the joints, it was found that they nowhere leave the middle third except at the right springing joint (17) where the centre of pressure was 0.08 ft. above the middle third limit. Hence it was thought best to increase the depth of keystone three times this amount, or 0.25 ft., *making the radial depth of arch ring uniformly* 2.75 *feet.*

This is so near the former value, 2.5 ft., that it was not thought worth while to test it by another construction. It will be assumed to satisfy all conditions.

The intensity of thrust at the upper edge of the right springing joint (for the arch ring 2.5 ft. deep) is found to be 9.2 tons per square foot, at the lower edge of the left springing joint 8.7 tons.

43. *Example* III. Segmental stone arch of 50 feet span, 10 ft. rise, radius 36.25 ft., and height of surcharge above crown 2 feet. A construction for a depth of keystone of 3 feet showed, for the loading to be given, that the line of resistance passed outside the middle third; hence, for a second trial, a depth of arch ring of 3.5 feet was assumed. The loading omitted the pilot wheel and consisted of the eight drivers on the left half of the arch, viz.: 15 tons 4.25 feet from crown; 15 tons 10 feet; 15 tons 14.25 feet; and 15 tons 19 feet, all to left of crown of arch.

The line of resistance nowhere passed outside the middle third of the arch ring. At the springing joints the resultants touch the lower middle third limit on the loaded side; and the upper limit on the unloaded side; at the crown the thrust passes through the centre of the crown joint. The arch thus satisfies all the conditions of stability. The horizontal component of thrust at crown = 279.3 cu. ft. = 19.55 tons, and the vertical component = 21 cu. ft. = 1.47 tons. The normal component of the thrust at the left springing = 413.5 cu. ft. = 28.95 tons, giving an intensity at the intrados of

$$2\,\frac{28.95}{3.5}=16.5 \text{ tons},$$

which is within the proper limit.

This arch then satisfies all conditions for this loading.

44. *Example* IV. Stone arch of 100 ft. span, 20 ft. rise, radius 72.5 ft., depth of keystone 5 ft., and height of surcharge above crown 2 ft. The loading consisted of one locomotive and tender, as specified in Art. 34, covering the left half of arch, the right half being unloaded. The pilot wheel was placed 8 feet to left of crown, the other wheels following in order at the distances given in Art. 35, so that the last tender wheel was barely on the arch.

The line of resistance was everywhere contained within tho middle third, except at the left springing joint where it passed 0.17 ft. below the limit; hence to satisfy the middle third limit the arch ring should be increased in depth, say $3 \times 0.17 = .51$ ft., making the depth 5.5 ft.

The thrust at the crown joint fell 0.4 ft. below centre for the arch of 5 ft. key, its horizontal component being 689.7 cu. ft. = 48.279 tons, and the vertical component 26.5 cu. ft. = 1.855 ton. The intensity of thrust at lower edge = 14.3 tons per sq. foot.

The thrust at the left springing acted 1 foot below the centre of the joint, its normal component being 1048 cu. ft. = 73.36 tons. It acts as a uniformly increasing stress over a depth of joint = $3 \times 1.5 = 4.5$ ft.; hence the average stress is $73.36 \div 4.5 = 16.3$ and the intensity at lower edge is double this, or 32.6 tons per square foot. For a 5.5 ft. key the intensity is much less.

45. *Example* V. The arch of Ex. IV. of 100 ft. span, 20 ft. rise, and 5 ft. depth of key, subjected only to its own weight.

Here we need consider only the right half of the arch, since the thrust at the crown, from consideration of symmetry, must be horizontal as well as the line m m′ (using the designation given on plate for another span).

Hence for an assumed horizontal thrust at the crown,

having found points such as b and drawn the line k k′ as before, we determine line m m′ for points b exactly as we found k k′ for points a; or it is generally shorter to add up with dividers the ordinates to points b above a trial line as k k′, and subtract from the sum the length we obtain by adding up ordinates to points b below k k′. The difference divided by 8 gives the amount the horizontal m m′ is above or below k k′ to satisfy the condition $\Sigma\ \overline{(mb)} = 0$. The points b meant above are $b_9, b_{10}, \ldots . b_{16}$. As a test the sum of the ordinates from m m′ to points b above should exactly equal the sum to points b below m m′.

The sum of the products $\Sigma\ \overline{(ka\ .\ y)}$ is made out exactly as before. Its value for the half arch in this case is 302.3. Similarly find $\Sigma\ \overline{(mb\ .\ y)} = mb_9\ .\ y_9 + mb_{10} . y_{10} + mb_{11} . y_{11} + mb_{12}\ .\ y_{12} + mb_{13}\ .\ y_{13} - (mb_{14},\ y_{14} + mb_{15}\ .\ y_{15} + mb_{16}\ .\ y_{16}) = 388$, the ordinates mb being measured from m m′ up (+) or down (—) to points b.

On diminishing all the ordinates mb in the ratio of 302.3 to 388 and laying them off from k k′, we find the points c through which the resultants on the joints pass.

The point of thrust at the crown is found as usual, and the new pole is found by laying off on a horizontal through C' a distance equal to old pole distance $\times \frac{388}{302.3}$. The points c can now be tested and the resultants produced to intersection with all the joints.

It is interesting to compare the new curve of the centres of pressure for the bridge unloaded, with that found previously for the load on the left half. Thus remembering that the depth of arch ring is 5 ft., one-sixth of which is 0.83 ft. from the centre to curves defining middle third limits, the following tables give the distance *measured along any joint* from its centre to the centre of pressure of the joint, plus distances being measured upwards, minus distances downwards. The upper numbers, under the joint numbers, correspond to the arch unloaded and the lower numbers to the arch loaded.

RIGHT SIDE.

Crown	9	10	11	12	13	14	15	16	17
—.27	—.30	—.20	0	+.20	+.35	+.32	0	—.22	—.40
—.40	—.57	—.62	—.56	—.38	—.15	+.05	+.10	+.22	+.15

LEFT SIDE.

8	7	6	5	4	3	2	1	0
—.30	—.20	0	+.20	+.35	+.32	0	—.22	—.40
—.26	+.18	+.60	+.75	+.60	+.30	+ 13	—.65	-1.00

It will be observed at joints 0, 5 and 10, where the centres of pressure for bridge loaded are farthest from centre of joints, that when the live load comes on the centres of pressure remain on the same side of the centre line of the arch ring as for bridge unloaded. At many other joints as 17 the reverse obtains. Hence, if the arch was not circular, but of such a figure that, for bridge unloaded, its centre line would be the focus of the centres of pressure on the joints, the departure of the line of resistance for bridge loaded at joints 0, 5 and 10 would not be so great as above, though at joint 17 (which is often a critical joint), and at some other points, it would be greater. Therefore such a design would often permit of smaller arch rings, such that the line of resistance for the bridge loaded in any way could still be inscribed in the middle third. However, in some of the bridges examined (Exs. II. and III.) the centre of pressure on joint 17, for bridge loaded, passed through the upper middle third limit, so that if an arch having its centre line, the line of resistance for arch unloaded, was used here of the same depth as before, the centre of pressure on joint 17 would leave the middle third, and the arch would not be as stable as before.

As we cannot tell, without a special investigation of this kind, which design will prove the most economical, it is well to hold on to the *segmental* circular arch until the others, *for all kinds of loading*, are *proved* the most econom

ical, particularly as it is much easier to construct. and the economy, if any, in replacing it by the other, must be small. Writers generally, in advocating the catenarian curves, have not properly considered the preponderating influence of heavy eccentric loading.

46. An examination of the lines of resistance in all the preceding examples fails to indicate any simple approximate rule for constructing them without recurring to the theory of the solid arch. I have stated elsewhere, partly on the strength of a few constructions after the theory of the solid arch, for uniform loads or comparatively light eccentric loads, that "it seems highly probable that the *actual* line of resistance is confined within such limiting curves, approximately equidistant from the centre line of the arch ring that only one line of resistance can be drawn therein." From the constructions above this rule is found to indicate very roughly about the position, but it is not precise enough in practice. Thus for the 100 ft. span above unloaded, the true curve passes .4 below the centre line (measured, not vertically, but along the joint) at the springs, .35 above at joints 4 and 13 and .27 below at the crown;

but the divergences are much greater at the joints of rupture (0, 4 or 5, 9 and 17) for the arch heavily loaded on one side, as we see from the table, and the same thing is shown on the plate for the 12.5 feet span.

Hence we cannot state precisely that if a line of resistance can be inscribed in the middle third, the true line of resistance will be found in the middle third. It is in fact plain from the above constructions that if only *one* line of resistance can be inscribed in the middle third, the true line will pass outside of it at certain points; for the first line *touches* the curves limiting the middle third at all the joints of rupture, as it corresponds to both the maximum and minimum of the thrust within those limits, whereas the true curve does not at all the joints, hence it cannot agree with the former and hence must lie outside the middle third limits, since by assumption only one line of resistance can be drawn therein.

We can appropriately quote here Winkler's notable theorem published in 1879,

which is practically true for segmental arches of constant cross-section. The theory is given in full in an article by Prof. Geo. F. Swain on the stone arch, in Van Nostrand's Magazine for October, 1880.

Winkler's theorem is as follows:

That line of resistance is approximately the true one which lies nearest the centre line of the arch ring as determined by the method of least squares.

This remarkable theorem is easily demonstrated by aid of the theory of elasticity, and while it is no aid practically in precisely fixing the true line of resistance, yet the conclusions are valuable as confirming, in a general way, the preceding constructions.

47. The method of finding the true resistance line given in this chapter is perfectly general and applies to any *form of arch* of constant cross-section. The defect in the method practically is that the most hurtful position of the live load cannot be readily ascertained. It is true that for single loads the method of this chapter will give the quantities c_1, y and c_2 as defined

in Chap. IV.,* from which we may make out a table and proceed as in the preceding chapter. This method is very long and is rarely needed, as circular arches are generally built; and for these the quantities c_1, y and c_2 can be readily found from existing tables and formulas.

The positions of live loads assumed in this chapter simply followed the rough rule of putting the heaviest part of the load over the middle of the haunches. The constructions resulting were all made before the method of the preceding chapter was developed. It is gratifying to note that the conclusions as to depth of key, etc., are almost identical with those of Chap. IV., where the most hurtful position of the live load was carefully ascertained.

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* See such constructions in "Theory of Solid and Braced Elastic Arches," p. 79.

APPENDIX.

The writer, in 1874, performed some experiments on light wooden arches at the limit of stability, which tend to confirm theory and are instructive in many ways. They will be given in full below. The experiments were made with great care; the voussoirs being accurately cut, the span kept invariable and horizontal, piers vertical, and the weight applied very gently and without shock.

To avoid mistake the following momenclature will be adopted:

Depth of a voussoir is the dimension in the direction of the radius of the intrados $\perp$ to the axis of the arch.

Thickness of an arch is the dimension $\parallel$ to the axis of the arch.

Width of a pier is its horizontal dimension $\perp$ to the axis of the arch.

Height is measured vertically.

The dimensions will all be given in inches.

A Gothic arch (Fig. 28) of 14 in. span, and 12.12 in. rise, was cut out of a poplar (tulip tree) plank, 3.65 in. thick, consisting of 8 voussoirs, each 3.65 thick, 2 deep, and 4.08 along their centre line from middle to middle of joint; each voussoir weighing .52 lb. Quite a number of voussoirs were cut out of the same layers of fibres and those selected that weighed exactly the same: the voussoir to be tried being hung to one end of a delicate balance beam, with a voussoir of the standard weight at the other end. The two voussoirs at the crown not being cut out of the same layers of fibres as the others, were shaved off about the middle of the extrados (not touching the joints) so as to weigh exactly 1 voussoir of the standard weight and their centres of gravity were found experimentally, and found to be at exactly similar points in both voussoirs, so that the entire arch was symmetrical as to the crown.

The centres of gravity of the other voussoirs are taken on the arc of a circle passing through the middle of the joints, and for any voussoir, equidistant from the joints bounding that voussoir. For voussoirs whose sides are

little inclined this is sufficiently near the truth, and by dividing the arch ring into a sufficient number of artificial voussoirs the result may be made as accurate as we please. Still as no wood is homogeneous the results can only be regarded as approximate as compared with the hypothetical homogeneous arch, still sufficiently near to establish the laws heretofore demonstrated.

When this arch was set up the joints apparently fitted perfectly, and on placing a drawing-board by the side of the arch and tracing off its contour curves, it was found to be a perfect Gothic whose arcs, composing the contour curves were correct arcs of circles described from the springing points opposite.

A number of rectangular wooden bricks of exactly 1 voussoir in weight, of various sizes, were also cut out, as well as half bricks, quarter bricks, etc., and some solid rectangular piers of various dimensions.

A voussoir is taken as the unit of weight.

In experiments where weights were placed upon the top of the arch, an assistant added brick after brick, carefully balancing the load at the top on either side by the fingers until the arch reached its *balancing point; i. e.*, the point where it stood with the weight, but fell with a slight jarring.

The two bottom voussoirs were, when necessary, kept from sliding by two fastening tacks being driven into the board on which the arch rested, pressing against the arch .03 above the springing line, or so little that it may be disregarded. The board was carefully levelled at every experiment by a spirit level, and the span kept invariably at 14 in.

There was little or no vibration in the room.

First Experiment.—With 8.2 voussoirs on the summit of the arch it stood, though fell with 8.3 voussoirs on the summit; rotating on joints 2 on intradoal edge, joints 4 at the extrados and at the upper edge of the crown joint, the arch being forced out at the haunches and falling at the crown. (See Fig. 28.)

The following table gives in its first column the number of joint from the crown; column *s*, the *elementary weights* (4.1 voussoir being the weight on the summit that goes to each abutment, the weight of each voussoir being taken as unity); column *m* gives the horizontal distance from the crown to the centre of gravity of each voussoir with its load, if any, which, in this case, is also the *moment* in reference to the vertical through the crown of each voussoir. Columns S, M, and C have been before explained:

	s	m	S	M	C
	4.1	0.00	4.1	0,00	0.08
1	1.	1.70	5.1	1.70	.33
2	1.	4.68	6.1	6.38	1.04
3	1.	6.79	7.1	13.17	1.85
4	1.	7.88	8.1	21.05	2.6
	8.1	21.05			

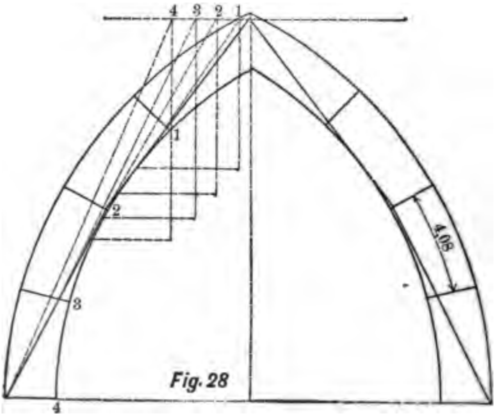

Fig. 28

Try a line of resistance, passing 0.1 from the upper edge of crown joint and 0.1 from the extrados edge of the joint at the springing. It is found to cut joint 2 at 0.1 from the intrados.

From joint 0 to joint 2 the line of pressures corresponds to the minimum of the trust; from joint 2 to joint 4, to the maximum within limiting curves 0.1 from intrados and extrados respectively. (Art. 20.)

Sliding would have occurred on joint 4, as the resultant on that joint made an angle of 24 deg. with the normal, but for the tacks before mentioned.

The diagrams for this and all the following experiments were drawn to a scale of one-third the natural size, except in the case of some of the pier experiments.

It may pertinently be inquired, why at the limit of stability, the centres of pressure should not be found at the very edges of the joints in place of being 0.1 inch from those edges? The answer is simple: We have seen in Art. 21 that when the centre of pressure on a joint leaves the middle third, the joint begins to open, and this opening is quite perceptible when this centre of pressure is very near the edge. This opening of the joints causes a deformation of the arch ring, so that the figure just before rotation occurred is not that assumed in the drawing. If the deformation had been known at the instant of rupture, so that the true figure could have been drawn, then the line of resistance would have passed through the very edges of the joints 0, 2 and 4, as they alone were bearing at the time. No attempt was made to find the deformed figure; in fact, it varied so rapidly just before rupture that it would have been impossible to have found it. Similar remarks and explanations apply to all the subsequent experiments.

Second Experiment.—With the two voussoirs at the crown in one solid piece, the arch could not give by rotation, as the lower edge of crown joint could not open. With a sufficient pressure on the crown, there was sliding along joints 1, the coefficient of friction being small for these wooden bl cks.

We evidently have here a sufficient reason for making the keystone in one solid piece.

Third Experiment.—On placing a knife edge against a notch .03 deep, cut into the bottom voussoir, 0.4 above the springing line, on each side, the arch balanced with 11.1 voussoirs on the summit. The line of resistance must now pass through the knife edges, and it will be found on constructing a diagram that it will pass about 0.1 from edges at joints 0 and 2, as before.

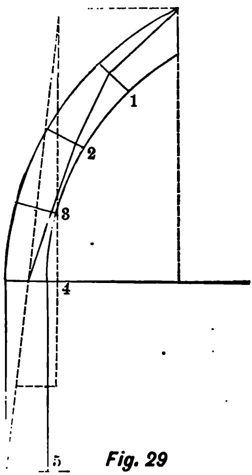

Fig. 29

Fourth Experiment—(Fig. 29.) The same arch stood, being very nearly on the balancing point, on *solid* piers 10. high. 1.9 wide, and 3.65 thick, each pier weighing 2.3 voussoirs, the intrados at the springing being at the inner edge of pier. The piers were made vertical by a spirit level, and their tops were upon the same level in every experiment given.

In the following table the pier is included opposite joint 5 of the first column:

	S	c	m	S	M	C
1	1	1.7	1.70	1	1.70	1.7
2	1	4.68	4.68	2	6.38	3.19
3	1	6.79	6.79	3	13.17	4.39
4	1	7.88	7.88	4	21.05	5.26
5	2.3	7.95	18.28	6.3	39.33	6.24
	6.3		39.33			

A line of resistance 0.1 from edges of joints 0 and 3, cuts the base of the pier 0.2 from its outer edge.

Fifth Experiment.—With piers 40.47 in. high, 3.65 wide, and 1.9 thick, weighing 10.1 voussoirs each, with the intrados of arch at springing on a line with inner edge of pier, the same arch balanced. The pier was built of a solid block 22 in. high and 5 bricks placed on top, one above the other to make up the 40.47 in height.

A line of resistance drawn .1 from summit and .1 from intrados at joint 3 passes .5 from outer edge of pier, or about 1-7 width of pier.

If the figure of the deformed arch could have been drawn at the instant of rupture the line would have passed through the very edges.

Sixth Experiment.—The pier of *Exp.* 4 (Fig. 29) was moved outward (*from* the axis of the arch) so that when its inner edge was .1 from the springing, it stood with no weight on the summit; when it was .4 from edge, it stood with .5 vs., fell with .6 vs.; .5 from edge, balanced with .75 .vs.; .6 from edge balanced with .75 vs.; .7 from edge balanced with .37 vs.; 1.0 from edge balanced with .12 vs.

On constructing the table and diagram as above for the load .75 vs., we find the centre of pressure on joint 4, .63 from the inner edge, or slightly over the extreme limit above, as should be the case.

Seventh Experiment.—The same arch stood easily with .75 vs. on the summit, on solid piers, 22. high, 3.65 wide, and 1.9 thick, each weighing 5.1 vs.; the arch fell with the addition of .12 vs. more.

On constructing this figure it will be found that the line of centres of pressure, assumed 0.1 from edges of joints 0 and 3 as before, passes .63 from inner edge of springing joint (as was stated above) and cuts the base of pier .39 from its outer edge or about 1-9 the width of pier.

Eighth Experiment.—On moving this pier back as in the 6th *Exp.*:

0.47	the	arch	balanced	with	1.12 vs.
0.53	"	"	"	"	1.25 "
.59	"	"	"	"	1.25 "
.63	"	"	"	"	1.25 "
.7	"	"	"	"	1.12 "
1.	"	"	"	"	1.00 "

On constructing the line of resistance for a weight of 1.25 at the apex, passing 0.1 from the edge of joints 0 and 3 as before, it will be found that the centre of pressure on joint 4 is .7 from the edge, again slightly over the extreme limit .63 found by experiment.

It is evident from an inspection of the arches in churches that constructors were well aware that a higher pier might be used when its inner edge was moved back a certain distance from the springing, which is equivalent to what we have established above.

Ninth Experiment.—With the pier used in *Exp.* 4. and the same arch, excepting that the two voussoirs at the crown were in one piece, the arch and pier just balanced as in *Exp.* 4. In fact the arch and pier can easily rotate on the third joint and the outer edge of pier.

Tenth Experiment.—The same arch with piers 1.98 wide, 7.5 high and thickness of arch, each weighing 2 vs., stood easily when a cylindrical pin .03 in diameter was placed at the lower edge of crown joint. This joint bore at no other point, hence the line of resistance passes through the pin. Assuming it to pass .1 from the edge of joint 3, the construction will show that it cuts the springing joint .6 from inner edge and the base of pier .15 from its outer edge.

The *experiments* that we have just considered very

clearly indicate the fallacy of that theory which supposes that if a line of resistance passes outside the inner third of the arch ring, *that it must fall.* On the contrary, in every case of the stability of the arches previously given, it is *impossible* to draw a line of resistance everywhere contained within the inner third of the arch ring.

Eleventh Experiment.—Fig. 30. With this same Gothic arch a segmental circular arch was now made of 24.24 in. span and 7 in. rise; the voussoirs being as before 2. deep and 3.65 thick.

With 7.6 vs. on the summit, this arch *balanced;* the weight being placed on a small stick resting on the summit. With a greater weight the rotation occurred on joints 0, 2 and 4, the crown falling.

	s.	m.	S.	M.	C.
	3.8	0.00	3.8	0.00	.00
1	1.	2.03	4.8	2.03	.42
2	1.	5.90	5.8	7.93	1.37
3	1.	9.38	6.8	17.31	2.55
4	1.	12.11	7.8	29.42	3.77
	7.8	29.42			

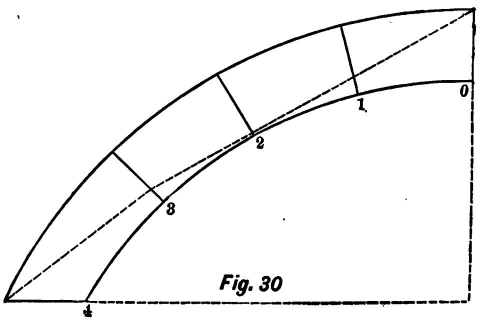

Fig. 30

On trial it was found that the true line of resistance passes .15 from the edges at joints 0, 4 and 2; giving the characteristics of both a maximum and a minimum thrust.

The ends of this arch required fastening tacks thrust into the board and pressing against voussoirs 4, .03 above the springing as in the first exp., with the Gothic. to prevent sliding. The thrust on joint 4 made an angle of 50° with the normal to that joint.

Twelfth Experiment.—With this arch resting on piers 3 63 wide, 5.8 high and 2. thick, each weighing 1.5 vs., the inner edge of pier being on a line with the springing, the arch balanced with .5 vs. on the summit.

We find, by constructing a line of resistance passing .15 from summit and the intrados at the third joint, that it cuts the base of pier .24 from its outer edge.

Thirteenth Experiment.—To form some idea of the action of mortar of different degrees of hardness, pieces of cloth .07 thick when not pressed, and .04 thick when pressed between two flat surfaces by the hands were put between the joints of the Gothic arch (Fig. 28), each piece weighing .015 voussoir.

The span was then altered until the joints were all close, when it was found to be 14.57, the rise to the apex being 14.55. On placing a drawing-board by the side of this arch and tracing its contour curves, they were found to be very nearly arcs of circles, though not with their centres at the springing points. To locate them; measure horizontally from the springing points .32 towards the middle of the span, and then vertically downwards 0.1 to the centres, from which the arch may be drawn.

This arch balanced with 4.6 vs. at apex; fell with 4.65 vs. The limiting lines to the curve of resistance was found to be distant .3 = 1-7 depth of joint from the contour curves, at its nearest approach to them.

This arch spread outwards upon the application of the weights, joint 2 being the point of rupture at the haunches; hence it is evident that if there had been a solid spandrel, or in this case, simply the pressure of the hands, to resist

this spreading, that the arch would not have fallen. The spandrel would have supplied horizontal forces in addition to the vertical ones due to its weight.

If the spandrel were not solidly built, at least up to joint 2, there would necessarily be derangement of the arch.

The curves of resistance were drawn in all the foregoing experiments, not taking into consideration the last mentioned derangement of the arch, which would have caused this curve to pass nearer the edges or exactly through them.

In fact, in most of the experiments, just before rotating, the edges alone seemed to be bearing. In the case of the simple Gothic, without cloth joints, when a sufficient weight was applied at the summit, the joint there and joint 2 opened sensibly before the balancing weight was put on. The segmental arch flew out at the second joints, falling at the crown, only opening when near the balancing point.

Isolated weights applied at the summit do not occur in practice, and it is hardly probable that a well-built viaduct, whose intrados is a segment of a circle with thin joints, will spread appreciably after the mortar has well set; and this is necessarily a stronger form of arch than the semi-circular, elliptical, or hydrostatic, where the spandrel thrust is generally required to cause stability.

If the latter profiles are desired, let the depth of the voussoirs be increased towards the abutment, so as to keep the line of resistance within the proper limits of the arch ring, when the constructor will be assured of stability.

It certainly seems singular, that engineers should ever *recommend* an arch like the hydrostatic, which necessarily requires a very effective spandrel thrust to keep the arch from tumbling down.

The spandrels must in such cases be built with the same care used with the arch stones, thus increasing the *expense*, while really losing in *strength*.

Fourteenth Experiment.—In the joints of the same

Gothic arch, pieces of soft woolen cloth .15 thick when not pressed, and .1 when pressed hard between two bricks by the hands, were next inserted, each piece of cloth weighing .027 voussoir. The span, when the joints were close, was found to be 15 in.; rise to apex, 14.63. The centres for describing the contour curves were 1.07 in. from the springing points measured horizontally towards the middle of span.

This arch balanced with 2.3 vs. on the apex.

Assuming this arch to preserve its figure, the curve of resistance passes about one-fourth of the depth of joint from the edges at its nearest approach to them.

This experiment gives us some idea of the effect of thick plastic mortar joints or of uncentreing an arch with fresh mortar joints.

Fifteenth Experiment.—A Gothic arch of about half the dimensions of the first given in *Exp.* 1 was cut out, really before the arch we have just been considering.

It was not found to be symmetrical as to weight, one-half weighing 1-32 of the whole arch more then the other half. Still as arches in practice are unsymmetrical as to weight at least it will be interesting to know, that assuming this arch to be symmetrical, the curve of pressures passes .075 from the edges at joints of rupture, more especially with weights at the apex.

All the preceding experiments were repeated with this arch and the same laws approximately established.

In the experiment with the cloth joints the cloth was .05 thick not pressed; .04 when pressed hard by the hand. The curve of resistance was found to pass .1 from the edges at the joints of rupture, with a weight on the apex, and nearly so in the pier experiment with no weight on the apex.

Sixteenth Experiment.—The Gothic arch given by Fig. 28 will now be considered with an unsymmetrical load. A stout needle was thrust into the second voussoir from the crown on the right side, in the direction of a vertical through its centre of gravity, as represented in Fig. 31. With

a weight of 3.3 vs. on the top of the needle, the arch balanced, opening at summit and lower end of joint 1 on the right. The voussoir to which the weight was added would have slid if pins had not been thrust into the edges of its joints, thus supplying a force analogous to friction, though not interfering at all with rotation.

We now form the following tables: the first being condensed from the one referring to Exp. 3.

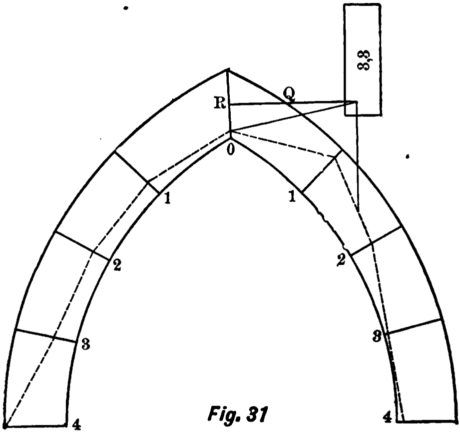

Fig. 31

LEFT SIDE.

	S	C
1	1	1.70
2	2	3.19
3	3	4.39
4	4	5.26

RIGHT SIDE.

	s	C	M	S	M	
1	1.	1.7	1.70	1.	1.70	1.70
2	4.3	4,68	20.12	5 3	21.82	4.12
3	1.	6,79	6.79	6.3	28.61	4.54
4	1.	7.88	7.88	7.3	36.49	5.
	7.3		36.49			

A line of resistance can be drawn, as shown in the figure, passing .18 from the extrados at joints 4 on left, and 1 on right and .18 from the intrados at joints 0 and 3 on the right.

The lower edge of the crown joint was imperfect, being the only imperfect edge in the arch, and this may account for the line of resistance retreating farther in the arch than for a load in the summit as before considered.

The thrust on joint 1, on the right, was inclined at an angle of 15° to the normal to that joint, which accounts for the sliding, as the joints were planed and across the grain.

Seventeenth Experiment.—The segmented arch, Fig. 30, was next tried with the eccentric load.

A short needle was thrust in voussoir 2 on the left, in the direction of the vertical through its centre of gravity, as shown in Fig. 32; the arch balanced with 5.4 voussoirs on the top of this needle.

We form the following tables:

RIGHT SIDE.

	s	m	S	M	C
1	1	2.03	1	2.03	2.03
2	1	5.90	2	7.93	3.96
3	1	9.38	3	17.31	5.77
4	1	12.11	4	29.42	7.37
	4	29.42			

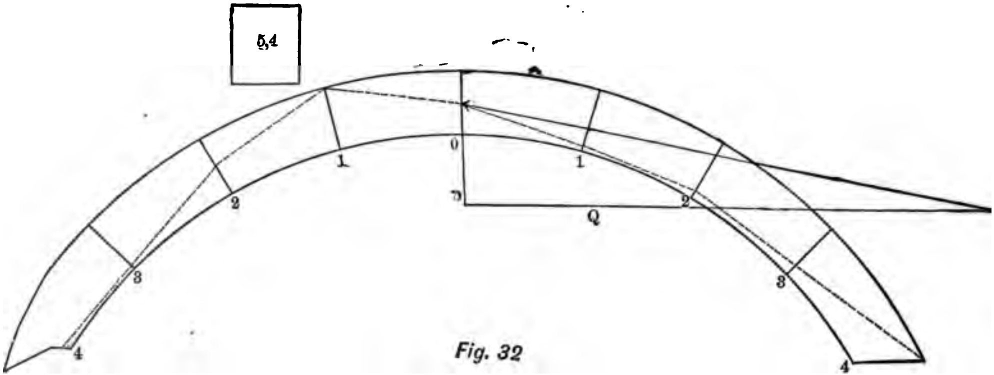

Fig. 32

LEFT SIDE.

	S	M	S	M	C
1	1.	2.03	1.	2.03	2.03
2	6.4	37.76	7.4	39.79	5.38
3	1.	9.38	8.4	49.17	5.85
4	1.	12.11	9 4	61.28	6.52
	9.4	61,28			

The voussoir on which the weight was placed would have slid along its joints but for pins being thrust into its edges in a manner that did not interfere with rotation.

A line of resistance was drawn that passes .15 from the intrados at joint 2 on the right and .2 distant from the edges at joints 4, 1 and 4; hence the true curve will probably pass about .18 from these edges. This is nearly (03 difference) what we obtained, for the limits from the edges of the line of resistance in the 11th Exp., Fig. 30. The thrust on joint 1 on the left is inclined 16° .5 to the normal to the joint, nearly what we found before. The sliding in this and the last experiment only occurred just before the balancing weight was applied; the line of pressures travelling down the crown joint as the weight was increased, until finally the direction of the pressure on joint 1 exceeded the complement of the angle of friction.

Eighteenth Experiment.—Figure 33 represents two rafters 9.92 in length, 1.9 width (dimension in plane of paper) and 3.60 thick, leaning against each other at the top and against piers 7.7 high, 1.98 wide and 3.6 thick at their bottom edge, which is moved back 0.6 from the edge of the pier. The horizontal distance between the vertical piers is 10 in., so that the feet of the rafters are 11.2 apart. Each rafter weighed 2.3 vs.; each pier 2. vs. The rafters and piers just balanced in this position.

Reasoning as in Art. 2, we see that the thrust at the upper edges of contact of the rafters is horizontal; hence draw a vertical line through the centre of gravity of the rafter equal to its weight; the resultant on the lower edge of the rafter passes through this edge, and combined with the

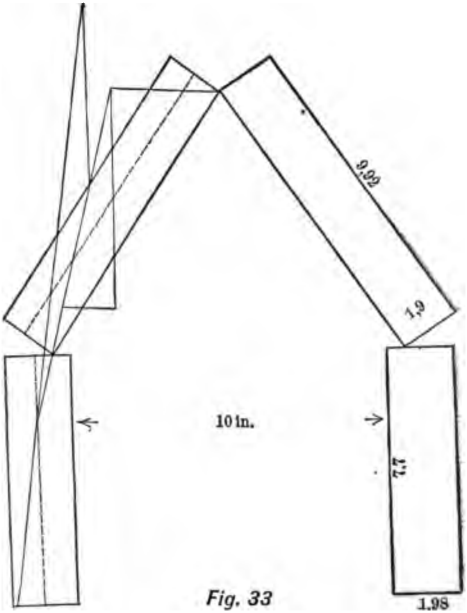

Fig. 33

weight of the pier acting through its centre of gravity, gives the resultant thrust on the base of the pier. In this case it strikes twenty-two hundredths (.22) from its outer edge.

This experiment was performed to ascertain whether the resultant on the base could ever be drawn through the outer edge of base for the original figure. It seemed probable, as the centres of pressure at the apex and top of the pier were absolutely fixed, and there was only one real

joint at the base of the pier; but we see, even from this case, that the joint opened sufficiently to deform the original figure, so that the resultant cannot be drawn exactly through the outer edge for the original figure. This should offer a valuable hint to experimenters and constructors, *not*

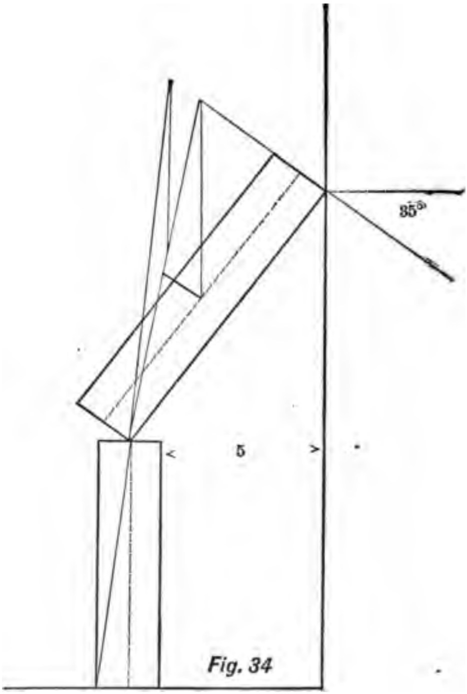

Fig. 34

to look for the stability in similar structures that the theory of "rigid" or incompressible bodies would give, especially structures composed of a great number of blocks without cementing material.

Nineteenth Experiment.—Fig. (34), represents a rafter and pier of the preceding experiment; the rafter leaning against a vertical rough plastered wall by its edge, the lower edge resting on the pier 1.03 back from its inner edge. This was the balancing position.

After several trials, assuming as we found in the preceding experiment, that the resultant strikes .22 from the outer edge of the base of pier, it was found that the direction of the thrust against the wall was inclined about 35° to the horizontal, which is about what we should imagine the angle of friction of the edge on the wall to be. If the thrust at the upper edge be assumed horizontal as is usual, it will be found that the final resultant passes outside the base of pier; hence, such an assumption is false. The construction (Fig. 34), will also show that .32 v. of the rafter is sustained by the wall, 1.98 v. being supported by the pier; *i. e.* about one seventh of the weight of the rafter is upheld by the friction of the plastered wall.

On leaning a half arch against a wall, it was found to balance on higher piers than when the other half was placed against it.

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